Model Theory and the Mordell-Lang´s conjecture

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Trabajo de Fin de Máster:

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Abstract

The main aim of this dissertation is to depict the most significant model-theoretic results related to the Mordell-Lang's conjecture and, in particular, to the Hrushovski's proof of this.

Therefore, after introducing the basic concepts of sorted languages, monster models and imaginaries, we study Morley's rank and the theory of groups with Morley's rank. Then, we apply these abstract results to algebraically closed fields and abelian varieties. Once this is done, we will study the Mordell-Lang conjecture and give a sketch of the Hrushovski's proof for the characteristic 0 case.

Resumen

El objetivo principal de este Trabajo de Fin de Grado es mostrar los resultados más importantes de teoría de modelos relacionados con la conjetura de Mordell-Lang y, en particular, con la demostración de Hrushovski de esta.

Por tanto, tras una introducción a los conceptos básicos de lenguages de varias clases, modelos monstruo e imaginarios, estudiaremos el rango de Morley y la teoría de grupos con rango de Morley. Entonces, aplicamos estos resultados abstractos a cuerpos algebraicamente cerrados y variedades abelianas. Una vez hecho esto, estudiaremos la conjetura de Mordell-Lang y daremos un esquema de la demostración de Hrushovski para el caso de característica 0.

Contents

Introduction

Model theory is a branch of mathematical logic which studies abstract mathematical objects (e.g., groups, rings, fields, vector spaces) as structures interpreting formal languages. Therefore, all in all model theory is considered an absolutely pure area with an almost insignificant number of useful applications by the whole of the mathematical community. However, it is far to be true. In the recent years, some relevant open problems from other areas have been solved with model theory. This memoir presents one of these cases: the Mordell-Lang's conjecture.

The Mordell-Lang's conjecture, made by Serge Lang, is a generalization of the Mordell's conjecture. The Mordell's conjecture, which was questioned by Louis Joel Mordell in 1922, states that a curve of genus greater than 1 over the field of rational numbers has only finitely many rational points. In 1983 and 1984, Gerd Faltings proved the Mordell's conjecture, which is now known as Faltings's theorem, using complex algebraic techniques. Faltings's theorem can be reformulated as a statement about the intersection of a curve with a finitely generated subgroup of an abelian variety. Generalizing by replacing the curve by an arbitrary subvariety and the finitely generated subgroup by an arbitrary finite-rank subgroup leads to the Mordell-Lang conjecture, which was also proved in the characteristic 0 case by Faltings (1991, 1994). Finally, the relative Mordell-Lang's conjecture is a generalization of this one to number fields of arbitrary characteristic. In 1996, Ehud Hrushovski proved the relative Mordell-Lang's conjecture using model theory. Thus, giving also a new proof for the characteristic 0 case.

The main aim of this dissertation is to study the model-theoretic content of Mordell-Lang's Conjecture and the model theoric background used in the Hrushovski's proof. The actual proof given by Hrushovski, of which we present only a sketch in the characteristic 0 case, goes beyond this project.

This document is divided into five sections, wherein we study the concepts of monster model, imaginaries, stable theories, Morley's rank and groups with Morley's rank. Finally, we apply them to the conjecture in the theorem 5.7 and in the sketch 5.3 at the end.

We end this introduction briefly mentioning the content of each chapter, for a more detailed description see the introduction in each chapter. The first chapter is a short introduction of notation and concepts. Actually, it is a generalization of the theory studied in the course Curso Avanzado de Álgebra of this Master's degree to sorted-languages. Also, at the end of the chapter, we introduce the concepts of monster model and imaginaries, and we prove some basic results about them. The second chapter is the main one of this dissertation. In it, we study the concept of Morley's rank, its relation with stable theories and the most fundamental results about it. The third chapter studies definable groups with Morley's rank, which are a good example of the utility of model theory in other areas. We apply the abstract theory studied in the memoir to algebraically closed fields in the forth chapter. The last one is dedicated to the conjecture. The majority of this dissertation is based on [?]. Theorem 3.22 is from [9], and lemma 5.6 of theorem 5.7 is from [10]. For the Hrushovski's proof, see [12].

1 Basic concepts and notations

In this chapter we introduce the notation and basic concepts we are going to use in the rest of the dissertation. Also, we state some fundamental results. The most significant concepts we are going to study are the following

1. Sorted-languages and structures: they are a straightforward generalization of first order languages and structures adding sorts for the symbols and elements. A standard example is the language of groups actions which has two sorts: one for the group and one for the set.

2. Monster models: they are an asymptotic way to obtain saturated structures. Monster models are models of a theory such that every model is an elementary substructure of the monster model.

3. Imaginaries: we consider expansions of the languages and the structures adding element for the definable equivalence classes. Therefore, imaginaries allow us to define and work with quotient definable sets.

The most significant results are the Compacteness theorem [Theorem 1.11], Ryll-Nardzewski's theorem [Theorem 1.27], theorem 1.29 which we apply for monster models and theorem 1.32.

1.1 Many-sorted languages and structures

We work with many-sorted languages. Given a non-empty set S, a *many-sorted language L with sorts S*, shortened as *S-language*, is a set C_s of *constants* of sort s for each $s \in S$, a set $F_{(s_1,\dots,s_k,s)}$ of *function symbols* of sort s for each $(s_1,\dots,s_k,s) \in S^{k+1}$ and a set $R_{(s_1,\dots,s_m)}$ of *relation symbols* for each $(s_1,\dots,s_m) \in S^m$. Thus, a (non-sorted) first order language is a sorted language with just one sort. The set of variables for each sort is infinitely countable. Terms and formulas are defined in many-sorted languages as it is usual for non-sorted languages, but with coherence between the sorts. We write $\mathrm{Ter}\, L$ for the set of terms of L and $\mathrm{For}\, L$ for the set of formulas of L. Given $\overline{x}$ an n-tuple of variables, $n \in \mathbb{N}^*$, we denote $\mathrm{For}_{\overline{x}} L$ for the set of L-formulas with free variables in $\overline{x}$. In $\mathrm{For}_{\overline{x}} L$ we also fix the order of the variables. If we do not fix the variables, we denote $\mathrm{For}_{\overline{s}} L$ for the set of formulas with at most n free variables of sorts $\overline{s} \in S^n$ and $\mathrm{For}_n(L)$ for the set of formulas with at most n free variables. We write $\mathrm{For}_0 L$ for the set of sentences. Given $\eta \in \mathrm{Ter}\, L$ or $\eta \in \mathrm{For}\, L$, we write $\eta(x_1,\dots,x_n)$ to indicate that the free variables of η are in $\overline{x}$ and we write $\eta(t_1,\dots,t_n)$ to indicate that $x_1,\dots,x_n$ have been replaced by $t_1,\dots,t_n$ simultaneously. If $L = \{C_s, R_{\overline{s}}, F_{\overline{s},s}\}_{s\in S,\overline{s}\in\bigcup^n S}$,

$$\mathrm{card}(L) = \sup\{\mathrm{card}(S), \mathrm{card}(C_s), \mathrm{card}(R_{\overline{s}}), \mathrm{card}(F_{\overline{s},s}) \;:\; s \in S, \overline{s} \in \bigcup{}^n S\}.$$

Note that $\mathrm{card}(\mathrm{For}\, L) = \max\{\aleph_0, \mathrm{card}(L)\}$.

Notation. In the rest of this memoir and except otherwise stated, L will denote an S-language.

Definition 1.1. **Structures**.- An *L-structure* $\mathfrak{A}$ is a pair $\left(A, (z^{\mathfrak{A}})_{z \text{ of } L}\right)$ such that $A = \{A_s\}_{s\in S}$ is a family of non-empty sets and

$$\begin{array}{ll} z \text{ constant of sort } s & z^{\mathfrak{A}} \in A_s \\ z \text{ function s. } f_{(s_1,\dots,s_k,s)} & z^{\mathfrak{A}}:\ A_{s_1} \times \cdots \times A_{s_k} \to A_s \\ z \text{ relation s. } R_{(s_1,\dots,s_m)} & z^{\mathfrak{A}} \subseteq A_{s_1} \times \cdots \times A_{s_m}. \end{array}$$

A is the *universe* of $\mathfrak{A}$ and $z^{\mathfrak{A}}$ is the *interpretation* of z in $\mathfrak{A}$. For many-sorted languages, a *sorted subset* is a family of subsets for each sort and a *sorted function* is a family of functions for each sort. A sorted subset is *finite* if both there are only finitely many non-empty sorts and there are finitely many elements of each sort. A sorted subset is *infinite* if there are infinitely many elements of each sort. We do a small abuse of notation by using the standard symbols for sets and functions for sorted sets and sorted functions, e.g., $a \in B$ for $a \in B_s$ where $s \in S$ is the sort of a, $f(a)$ for $f_s(a)$ where $s \in S$ is the sort of a, $B \subseteq C$ for $B_s \subseteq C_s$ for each $s \in S$, $\mathrm{card}(A) < \kappa$ for $\mathrm{card}(A_s) < \kappa$ for each $s \in S$ and $\mathrm{card}(A) > \kappa$ for $\mathrm{card}(A_s) > \kappa$ for each $s \in S$. We write $\mathfrak{A}_C$ to indicate an expansion of the L-structure $\mathfrak{A}$ by adding constants (or function or relation symbols) to the language L for a sorted subset C (or a sorted set of functions or relations). In that case, we write $L(C)$ for the corresponding expansion of the language. We say that $\mathfrak{A}$ is the *L-reduct* of $\mathfrak{A}_C$.

Definition 1.2. **Homomorphism**.- An *homomorphism* $\psi:\ \mathfrak{A} \to \mathfrak{B}$ between two L-structures is a sorted function from A to B such that:

$$\begin{array}{ll} c \text{ constant} & \psi(c^{\mathfrak{A}}) = c^{\mathfrak{B}} \\ f \text{ function s.} & \psi(f^{\mathfrak{A}}(a_1,\dots,a_k)) = f^{\mathfrak{B}}(\psi(a_1),\dots,\psi(a_k)) \\ R \text{ relation s.} & R^{\mathfrak{A}}(a_1,\dots,a_m) \Rightarrow R^{\mathfrak{B}}(\psi(a_1),\dots,\psi(a_m)). \end{array}$$

We define *embeddings*, *isomorphisms* and *automorphisms* in analogy to the non-sorted languages case. We write $\mathrm{Aut}(\mathfrak{A})$ for the set of automorphisms and $\mathrm{Aut}(\mathfrak{A}/B)$ for the set of automorphisms fixing the sorted subset B.

An *evaluation* in an L-structure $\mathfrak{A}$ is a family of functions $\vartheta = \{\vartheta_s\}_{s\in S}$ such that $\vartheta_s:\ V_s \to A_s$ for each $s \in S$, where V is the set of variables. Given an element $a \in A_s$, a variable $x \in V_s$ and an evaluation ϑ, we write $\vartheta, a/x$ for the evaluation defined by $\vartheta, a/x(x) = a$ and $\vartheta, a/x(y) = \vartheta(y)$ for $y \neq x$.

Given an evaluation ϑ in an L-structure $\mathfrak{A}$, we define the *interpretation* of terms t in $\mathfrak{A}$ by ϑ and the *satisfaction* of formulas φ in $\mathfrak{A}$ by ϑ in analogy to the non-sorted languages case, and denote them by $t^{\mathfrak{A}}[\vartheta]$ and $\mathfrak{A} \models \varphi[\vartheta]$ respectively.

Note that interpretations and satisfactions are independent from non-free variables, so we can write a finite tuple instead of ϑ. In particular, sentences are satisfied by one evaluation if and only if they are satisfied by every evaluation. Therefore, it make sense to say that an L-structure does or does not satisfy a sentence. Also, we have the following straightforward result:

Lemma 1.3. ***(Substitution lemma)*** *Let L be an S-language, $\mathfrak{A}$ be an L-structure, ϑ be an evaluation and $t_1,\dots,t_n \in \mathrm{Ter}\,L$ and $\varphi \in \mathrm{For}L$. Then,*

$$\mathfrak{A} \models \varphi(t_1,\dots,t_n)[\vartheta] \Leftrightarrow \mathfrak{A} \models \varphi\left[t_1^{\mathfrak{A}}[\vartheta],\dots,t_n^{\mathfrak{A}}[\vartheta]\right].$$

Definition 1.4. **Definable sets**.- Let $\mathfrak{A}$ be an L-structure and B be a sorted subset. A B-*definable* set is a set of tuples satisfying an $L(B)$-formula. A *definable* set is a B-definable set for some B. Given $\varphi \in \mathrm{For}_{\overline{x}} L(B)$, we write

$$\varphi[\mathfrak{A}/\overline{x}] := \{\overline{a} \in A_{s_1} \times \cdots \times A_{s_n} \ : \mathfrak{A}_B \models \varphi[\overline{a}/\overline{x}]\}.$$

We indicate the variables to fix an order, so we abbreviate $\varphi[\mathfrak{A}]$ when the order of the variables is clear from the context. Given a definable set D, we usually write $\underline{D}$ to indicate a formula defining D. Given a tuple $\overline{s} \in S^n$, we write $\mathrm{Def}^{\mathfrak{A}}_{\overline{s}}(B)$ for the boolean algebra of B-definable sets of $A_{s_1} \times \cdots \times A_{s_n}$. An element a is *definable* over a sorted set B if $\{a\} \in \mathrm{Def}^{\mathfrak{A}}_{s}(B)$, where s is the sort of a. We write $\mathrm{dcl}^{\mathfrak{A}}(B)$ for the sorted set of definable elements. An element a is *algebraic* over a sorted set B if there is a finite definable set $D \in \mathrm{Def}^{\mathfrak{A}}_{s}(B)$ with $a \in D$, where s is the sort of a. We write $\mathrm{acl}^{\mathfrak{A}}(B)$ for the sorted set of algebraic elements.

If V is a set of tuples from A, its algebraic closure $\mathrm{acl}(V)$ and its definable closure $\mathrm{dcl}(V)$ are respectively the algebraic closure and the definable closure of the sorted set of coordinates of the elements of V.

1.2 Theories

An L-*theory* is a set of sentences of L. A *model* of an L-theory T ($\mathfrak{A} \models T$) is an L-structure $\mathfrak{A}$ which satisfies every sentence of T. A *satisfiable* L-theory is an L-theory with models and a *finitely satisfiable* L-theory is an L-theory such that any finite subset is satisfiable. A sentence $\varphi \in \mathrm{For}_0(L)$ is a *consequence* of a theory T if every model of T satisfies φ. *Equivalent* theories are theories with the same models. The theory $\mathrm{Teo}(\mathfrak{A})$ of an L-structure $\mathfrak{A}$ is the set of all L-sentences satisfied by $\mathfrak{A}$. *Equivalent* L-structures are L-structures with the same theory. Note that isomorphic L-structures are equivalent. A *complete* theory is a theory such that all its models are equivalent. In particular, the theory of an L-structure is complete. The *atomic diagram* $\mathrm{Diag}(\mathfrak{A})$ of an L-structure $\mathfrak{A}$ is the set of atomic or negations of atomic $L(A)$-sentences satisfied by $\mathfrak{A}$.

Next we state a series of basic results for many-sorted languages whose proofs are analogous to the corresponding ones for non-sorted languages.

Lemma 1.5. ***(Models of the atomic diagram)*** *Let $\mathfrak{A}$ be an L-structure and $\mathfrak{B}$ an $L(A)$-structure with L-reduct $\mathfrak{B}'$. Then, $\mathfrak{B} \models \mathrm{Diag}(\mathfrak{A})$ if and only if $\psi : \mathfrak{A} \to \mathfrak{B}'$ defined as $\psi(a) = a^{\mathfrak{B}}$ is an embedding.*

An *elementary map* $f : C \to D$ is a sorted function between two sorted subsets C and D of two L-structures $\mathfrak{A}$ and $\mathfrak{B}$ which preserves satisfactions, i.e.,

$$\mathfrak{A} \models \varphi[\vartheta] \Leftrightarrow \mathfrak{B} \models \varphi[f \circ \vartheta]$$

for any evaluation ϑ in C.

An *elementary embedding* $f : \mathfrak{A} \to \mathfrak{B}$ is an embedding of L-structures which is an elementary map. An *elementary substructure* $\mathfrak{B}$ of an L-structure $\mathfrak{A}$ is a substructure such that the inclusion $\iota: \mathfrak{B} \to \mathfrak{A}$ is an elementary embedding, and then write $\mathfrak{B} \preceq \mathfrak{A}$.

Lemma 1.6. ***(Models of the theory)*** *Let $\mathfrak{A}$ be an L-structure and $\mathfrak{B}$ an $L(A)$-structure with L-reduct $\mathfrak{B}'$. Then, $\mathfrak{B}$ is model of* $\mathrm{Teo}(\mathfrak{A}_A)$ *if and only if $\psi: \mathfrak{A} \to \mathfrak{B}'$ defined as $\psi(a) = a^{\mathfrak{B}}$ is an elementary embedding of L-structures.*

Theorem 1.7. ***(Tarski's test)*** *Let $\mathfrak{A}$ be an L-structure and B a sorted subset of $\mathfrak{A}$. Then, B is the universe of an elementary substructure of $\mathfrak{A}$ if and only if for any $s \in S$ and any formula $\varphi(x) \in \mathrm{For}_s L(B)$*

$$\mathfrak{A}_B \models \exists x \varphi(x) \Leftrightarrow \text{ there exists } b \in B_s \quad \mathfrak{A}_B \models \varphi(b).$$

Theorem 1.8. ***(Tarski's Chain Lemma)*** *Let $\{\mathfrak{A}_j\}_{j \in J}$ be an elementary directed family of L-structures. Then, $\bigcup_{j \in J} \mathfrak{A}_j$ is an L-structure and $\mathfrak{A}_{j_0} \preceq \bigcup_{j \in J} \mathfrak{A}_j$ for each $j_0 \in J$. Moreover, if $\mathfrak{B}$ is an L-structure such that $\mathfrak{A}_j \preceq \mathfrak{B}$ for every $j \in J$, then $\bigcup \mathfrak{A}_j \preceq \mathfrak{B}$.*

Let $C = \{C_s\}_{s \in S}$ be a family of pairwise disjoint sets which are disjoint from L and let $g: \bigcup_{s \in S} \mathrm{For}_s(L(C)) \to C$ be a one-to-one function. A *Henkin's $L(C)$-theory with witnesses C (by g)* is an $L(C)$-theory containing the set

$$\{(\exists x \varphi(x)) \to \varphi(g(\varphi(x))) \,:\, \varphi(x) \in \mathrm{For}_1(L(C))\}.$$

Lemma 1.9. ***(Henkin's lemma)*** *Let T be a finitely satisfiable L-theory. Then, there exists a family $C = \{C_s\}_{s \in S}$ and a finitely satisfiable Henkin's $L(C)$-theory T^H with witnesses C such that $T \subseteq T^H$.*

Lemma 1.10. ***(Lindembaum's lemma)*** *Let T be a finitely satisfiable L-theory. Then, there exists $\overline{T}$ a finitely satisfiable L-theory such that $T \subseteq \overline{T}$ and for every sentence φ of L either $\varphi \in \overline{T}$ or $\neg\varphi \in \overline{T}$. In particular, if $\Delta \subseteq \overline{T}$ is finite and $\Delta \models \varphi$ for φ sentence, then $\varphi \in \overline{T}$.*

Proof. Apply the Zorn's lemma to the set of finitely satisfiable theories extending T. $\square$

Theorem 1.11. ***(Compactness theorem)*** *Let T be a finitely satisfiable L-theory. Then, T is satisfiable.*

Proof. Apply lemmas 1.9 and 1.10 in this order and let $\overline{T^H}$ be the $L(C)$-theory obtained. $\overline{T^H}$ is a finitely satisfiable Henkin's $L(C)$-theory with witnesses C such that $T \subseteq \overline{T^H}$ and, for any $\varphi \in \mathrm{For}_0 L(C)$, either $\varphi \in \overline{T^H}$ or $\neg\varphi \in \overline{T^H}$. Define, for each $s \in S$, the relation $\sim_s$ in C_s as $c \sim_s c' \Leftrightarrow c \doteq c' \in \overline{T^H}$. It is clear that $\sim_s$ is an equivalence relation. Consider the $L(C)$-structure $\mathfrak{A}$ with universe $A = \{A_s = {}^{C_s}/_{\sim_s}\}_{s \in S}$ and interpretation

constant c	$c^{\mathfrak{A}} = [c]$
function s. f	$f^{\mathfrak{A}}([c_1], \ldots, [c_k]) = [c_0] \Leftrightarrow f(c_1, \ldots, c_k) \doteq c_0 \in \overline{T^H}$
relation s. R	$R^{\mathfrak{A}}([c_1], \ldots [c_m]) \Leftrightarrow R(c_1, \ldots, c_m) \in \overline{T^H}$.

$\mathfrak{A}$ is well defined and $\mathrm{Teo}(\mathfrak{A}) = \overline{T^H}$. Hence, the L-reduct of $\mathfrak{A}$ is a model of T. $\square$

Corollary 1.12. *Let T be an L-theory and $\varphi \in \mathrm{For}_0 L$. Then, $T \models \varphi$ if and only if there is $\Delta \subseteq T$ finite such that $\Delta \models \varphi$.*

Theorem 1.13. ***(Löwenheim-Skolem-Tarski Theorems)*** *Let κ be a cardinal, $\mathfrak{A}$ be an L-structure and C be a sorted subset of $\mathfrak{A}$:*

1. ***(Downward)*** *If $\max\{\mathrm{card}(C), \mathrm{card}(L), \aleph_0\} \leq \kappa \leq \mathrm{card}(A)$, then there exists $\mathfrak{B}$, elementary substructure of $\mathfrak{A}$, such that $\mathrm{card}(B) = \kappa$ and $C \subseteq B$.*

2. ***(Upward)*** *If $\aleph_0 \leq \mathrm{card}(A) \leq \max\{\mathrm{card}(A), \mathrm{card}(L)\} \leq \kappa$, then there exists $\mathfrak{B}$, elementary extension of $\mathfrak{A}$, such that $\mathrm{card}(B) = \kappa$.*

Corollary 1.14. *Let T be a satisfiable L-theory which has an infinite model. Then, there exists a model $\mathfrak{B}$ of T for every cardinal $\kappa \geq \max\{\mathrm{card}(L), \aleph_0\}$ such that $\mathrm{card}(B) = \kappa$.*

Theorem 1.15. ***(Vaught test)*** *Let $\kappa \geq \mathrm{card}(L)$ be an infinite cardinal and T a satisfiable L-theory such that there is an isomorphism between any pair $\mathfrak{A}$ and $\mathfrak{B}$ of models of T with $\mathrm{card}(A) = \mathrm{card}(B) = \kappa$. Then, the infinite models of T are equivalent.*

Definition 1.16. **κ-categorical theories**.- Let $\kappa \geq \mathrm{card}(L)$ be an infinite cardinal and T a complete satisfiable L-theory. T is *κ-categorical* if T has models with infinitely many elements of each sort and there is an isomorphism between any two models of T with cardinal κ for every sort.

1.3 Saturation and types

Let $\mathfrak{A}$ be an L-structure, $\Sigma \subseteq \mathrm{For}L$ and ϑ be an evaluation in $\mathfrak{A}$. We say that ϑ *realizes* or *satisfies* Σ if $\mathfrak{A} \models \varphi[\vartheta]$ for every $\varphi \in \Sigma$, and then write $\mathfrak{A} \models \Sigma[\vartheta]$.

Definition 1.17. **Types**.- Let $\overline{x}$ be an n-tuple of variables, $n \in \mathbb{N}^*$, T an L-theory, $\mathfrak{A}$ an L-structure and B a sorted subset of $\mathfrak{A}$. A *$\overline{x}$-type with parameters B in* $\mathfrak{A}$ is a subset $\Sigma \subseteq \mathrm{For}_{\overline{x}} L(B)$ which is finitely satisfiable in $\mathfrak{A}_B$. An *$\overline{x}$-type of* T is a subset $\Sigma \subseteq \mathrm{For}_{\overline{x}}(L)$ such that $T \cup \Sigma$ is finitely satisfiable. A type $p(\overline{x})$ is *complete* if it is maximal, i.e., either $\varphi \in p(\overline{x})$ or $\neg\varphi \in p(\overline{x})$ for every $\varphi \in \mathrm{For}_{\overline{x}} L(B)$. A type is *global* if it is complete and its class of parameters is the whole universe. A *strong* type is a complete type with an acl-closed set of parameters.

We write $\mathbf{S}^{\mathfrak{A}}_{\overline{x}}(B)$ for the *space of complete $\overline{x}$-types* with parameters B in $\mathfrak{A}$ and $\mathbf{S}_{\overline{x}}(T)$ for the space of complete $\overline{x}$-types of T. We fix the variables for technical reasons, so when the variables are clear from the context we do not indicate these ones and write only $\mathbf{S}^{\mathfrak{A}}_{\bar{s}}(B)$ or $\mathbf{S}^{\mathfrak{A}}_{n}(B)$. Also, given $\varphi \in \mathrm{For}_{\bar{x}}\, L(B)$, write $\mathbf{S}^{\mathfrak{A}}_{\varphi}(B) \subseteq \mathbf{S}^{\mathfrak{A}}_{\bar{x}}(B)$ for the $\bar{x}$-types with parameters B *in* φ, i.e., the types in which φ is.

By the Zorn's lemma, given an L-theory T, for any type $\Sigma(\overline{x}) \subseteq \mathrm{For}_{\overline{x}} L$ of T there is a complete type $p(\overline{x})$ of T such that $\Sigma(\overline{x}) \subseteq p(\overline{x})$.

Also, for any tuple $\overline{a}$ from $\mathfrak{A}$, the *type of $\overline{a}$ over B in $\mathfrak{A}$* is

$$\mathrm{tp}^{\mathfrak{A}}_{\overline{x}}(\overline{a}/B) := \{\varphi(\overline{x}) \in \mathrm{For}_{\overline{x}}(L(\underline{B})) \,:\, \mathfrak{A} \models \varphi[\overline{a}/\overline{x}]\}\,.$$

It is clear that $\mathrm{tp}^{\mathfrak{A}}_{\overline{x}}(\overline{a}/B)$ is a complete type. When the variables and the structure are clear from the context we write $\mathrm{tp}(\overline{a}/B)$. Also, if V is a set of tuples, write $\mathrm{tp}(\overline{a}/V) := \mathrm{tp}(\overline{a}/B)$ where B is the sorted set of coordinates of elements of V.

If we substitute the variables of $\Sigma \subseteq \mathrm{For}L$ by new constants, we can apply the Compactness Theorem [Theorem 1.11] and conclude that Σ is satisfiable if and only if it is finitely satisfiable. Thus, given an L-theory, $\Sigma(\overline{x}) \subseteq \mathrm{For}_{\overline{x}}L$ is a type of T if and only if $T \cup \Sigma(\overline{x})$ is satisfiable. Then, since Σ is finitely satisfiable in $\mathfrak{A}$ if and only if $\Sigma \cup \mathrm{Teo}(\mathfrak{A}_A)$ is finitely satisfiable, we conclude that Σ is finitely satisfiable if and only if there is an elementary extension of $\mathfrak{A}$ where Σ is realized. Note that by Löwenheim-Skolem-Tarski theorems we can fix the cardinal of this elementary extension as $\kappa \geq \max\{\mathrm{card}(L), \aleph_0\}$ for every sort if $\mathrm{card}(A) \geq \aleph_0$.

Consequently, $\mathbf{S}^{\mathfrak{A}}_{\overline{x}}(B) = \mathbf{S}_{\overline{x}}(\mathrm{Teo}(\mathfrak{A}_B))$. In particular, if $\mathfrak{A}_B \equiv \mathfrak{A}'_B$, then $\mathbf{S}^{\mathfrak{A}}_{\overline{x}}(B) = \mathbf{S}^{\mathfrak{A}'}_{\overline{x}}(B)$. Also, if T is a complete L-theory, then $\mathbf{S}_{\overline{x}}(T) = \mathbf{S}^{\mathfrak{A}}_{\overline{x}}(\emptyset)$ for any $\mathfrak{A} \models T$.

We have a bijection

$$\begin{array}{rccc} F: & \mathbf{S}^{\mathfrak{A}}_{\overline{x}}(B) & \to & \left\{F \subseteq \mathrm{Def}^{\mathfrak{A}}_{\overline{s}}(B) \,:\, F \text{ ultrafilter in } \mathrm{Def}^{\mathfrak{A}}_{\overline{s}}(B)\right\} \\ & p(\overline{x}) & \mapsto & \{\varphi[\mathfrak{A}/\overline{x}] \,:\, \varphi(\overline{x}) \in p(\overline{x})\}\,. \end{array}$$

Thus, we define the topology of $\mathbf{S}^{\mathfrak{A}}_{\overline{x}}(B)$ via F by the *Stone's topology* of the boolean algebra $\mathrm{Def}^{\mathfrak{A}}_{\overline{s}}(B)$. This topology is defined by the base $\{\langle\varphi\rangle \,:\, \varphi \in \mathrm{For}_{\overline{x}}L(B)\}$ where $\langle\varphi\rangle := \left\{p \in \mathbf{S}^{\mathfrak{A}}_{\overline{x}}(B) \,:\, \varphi \in p\right\}$ for every $\varphi \in \mathrm{For}_{\overline{x}}L(B)$. Note that for any $\Sigma \subseteq \mathrm{For}_{\overline{x}}L(B)$, the set $\langle\Sigma\rangle = \bigcap_{\varphi\in\Sigma}\langle\varphi\rangle$ is a closed sets of $\mathbf{S}^{\mathfrak{A}}_{\overline{x}}(B)$.

Proposition 1.18. *Let $\mathfrak{A}$ be an L-structure and B a sorted subset. Then, $\mathbf{S}^{\mathfrak{A}}_{\overline{x}}(B)$ is a compact Hausdorff's space.*

Proposition 1.19. *Let $\mathfrak{M}$ and $\mathfrak{M}'$ be L-structures, $A \subseteq B$ sorted subsets of $\mathfrak{M}$ and $h:\ \mathfrak{M} \to \mathfrak{M}'$ an elementary embedding. Let*

$$\begin{array}{rccc} r: & \mathbf{S}^{\mathfrak{M}}_{\overline{x}}(B) & \to & \mathbf{S}^{\mathfrak{M}}_{\overline{x}}(A) \\ & p & \mapsto & p_{|A} := p \cap \mathrm{For}_{\overline{x}}L(A) \end{array} \quad \textit{and} \quad \begin{array}{rccc} h: & \mathbf{S}^{\mathfrak{M}}_{\overline{x}}(A) & \to & \mathbf{S}^{\mathfrak{M}'}_{\overline{x}}(h(A)) \\ & p & \mapsto & h(p). \end{array}$$

Then, r and h are continuous. Moreover, r is also an onto closed map.

Proof. It is clear that r and h are continuous. It is also clear that r is onto. Let us prove that r is a closed map. We have that the closed sets are of the form $\langle\Sigma\rangle$ where $\Sigma \subseteq \mathrm{For}_{\overline{x}}L(B)$. Then, it suffices to prove that $r(\langle\Sigma\rangle_{\mathbf{S}^{\mathfrak{M}}_{\overline{x}}(B)}) = \langle\Sigma'\rangle_{\mathbf{S}^{\mathfrak{M}}_{\overline{x}}(A)}$ where $\Sigma' = \{\varphi' \in \mathrm{For}_{\overline{x}}L(A) \,:\, \text{ there is } \varphi \in \Sigma\ \ \mathfrak{M} \models \forall\overline{x}(\varphi \leftrightarrow \varphi')\}$. The latter is clear. $\square$

By definition, any basic set $\langle\varphi\rangle$ is open and closed. Since boolean combinations of basic sets are basic sets, we conclude that an open and closed set is a basic set by compactness of $\mathbf{S}^{\mathfrak{A}}_{\bar{x}}(B)$.

Definition 1.20. **Saturation**.- Let $\mathfrak{A}$ be an L-structure and $\kappa \geq \operatorname{card}(L)$ an infinite cardinal. We say that $\mathfrak{A}$ is *κ-saturated* if any complete type $p \in \mathbf{S}^{\mathfrak{A}}_{x}(B)$ is realized in $\mathfrak{A}$, for every sorted subset B of $\mathfrak{A}$ with $\operatorname{card}(B) < \kappa$. We say that an L-structure $\mathfrak{A}$ is *saturated* if there is a cardinal $\kappa \geq \operatorname{card}(L)$ such that $\operatorname{card}(A) = \kappa$ and $\mathfrak{A}$ is κ-saturated.

Note that definable sets in κ-saturated structures are finite or of cardinal greater or equal than κ.

Remark. If $\mathfrak{A}$ is κ-saturated, every $p \in \mathbf{S}^{\mathfrak{A}}_{\bar{x}}(B)$ is realized, for every sorted subset B with $\operatorname{card}(B) < \kappa$ and every n-tuple of variables $\overline{x}$. We prove it by induction on n. Consider the type $\widetilde{p} = \{\exists x_2 \dots \exists x_n \varphi(x_1, \dots, x_n) : \varphi(x_1, \dots, x_n) \in p\}$, then $\widetilde{p}$ is realized, since $\mathfrak{A}$ is κ-saturated. Let $a_1 \in A_{s_1}$ be a realization of $\widetilde{p}$ and consider the type $p' = \{\varphi(a_1, x_2, \dots, x_n) : \varphi(x_1, \dots, x_n) \in p\}$. By hypothesis of induction there are $a_2, \dots, a_n$ such that $\mathfrak{A} \models p[\overline{a}/\overline{x}]$.

Definition 1.21. **Homogeneity**.- Let $\mathfrak{A}$ be an L-structure and $\kappa \geq \operatorname{card}(L)$ an infinite cardinal. We say that $\mathfrak{A}$ is *κ-homogeneous* if for every elementary map $f : B \to A$ there is an elementary extension $\widetilde{f} : B \cup \{a\} \to A$, for any sorted subset B with $\operatorname{card}(B) < \kappa$ and for any $a \in A$. We say that an L-structure $\mathfrak{A}$ is *homogeneous* if there is a cardinal $\kappa \geq \operatorname{card}(L)$ such that $\operatorname{card}(A) = \kappa$ and $\mathfrak{A}$ is κ-homogeneous.

Remark. A κ-saturated structure is κ-homogeneous. Indeed, let B be a sorted subset with $\operatorname{card}(B) < \kappa$ and a be an element. For any elementary map $f : B \to A$, $p = f(\operatorname{tp}_x(a/B))$ is a type over $f(B)$ and $\operatorname{card}(f(B)) < \kappa$, so there is an element a' realizing p. The extension to $B \cup \{a\}$ defined as $a \mapsto a'$ is an elementary map.

For $\kappa < \operatorname{card}(L)$, saying that an L-structure is κ-saturated or κ-homogeneous we mean that there is a $\lambda \geq \operatorname{card}(L) > \kappa$ such that it is λ-saturated or λ-homogeneous. Therefore, $\aleph_0$-saturated or $\aleph_0$-homogeneous do not mean that the language is countable.

Lemma 1.22. *Let κ be an infinite cardinal such that $\kappa \geq \operatorname{card}(L)$ and $\mathfrak{A}$ an infinite L-structure such that $\operatorname{card}(A) \leq 2^\kappa$. Then, there exists an elementary extension $\mathfrak{B}$ of $\mathfrak{A}$ such that $\operatorname{card}(B) \leq 2^\kappa$ and realizing every 1-type of $\mathfrak{A}$ with at most κ parameters of each sort.*

Proof. For every $s_0 \in S$, fix a variable x of sort s_0 and let $P_{s_0} = \{p_\xi(x)\}_{\xi\in\alpha_{s_0}}$ be the set of complete x-types with at most κ parameters for each sort. Note that $P_{s_0} = \bigcup\{\mathbf{S}^{\mathfrak{A}}_{x}(D) : D_s \in \mathcal{P}^{\leq\kappa}(A_s) \text{ for each } s \in S\}$, so $\operatorname{card}(P_{s_0}) \leq 2^\kappa$. Indeed, the cardinal of the set of sorted subsets with at most κ elements for every sort is $\prod_{s\in S}\operatorname{card}(A_s)^\kappa = 2^\kappa$ and, for any sorted set D with $\operatorname{card}(D) \leq \kappa$, we know that

$$\operatorname{card}\left(\mathbf{S}^{\mathfrak{A}}_{x}(D)\right) \leq \operatorname{card}\left(\operatorname{For}_x L(B)\right) = \max\{\aleph_0, \kappa, \operatorname{card}(L)\} = \kappa.$$

Let $C = \{C_s\}_{s \in S}$ be a sorted set of new constants such that $C_s = \{c_\xi\}_{\xi \in \alpha_s}$. Then, $T' = \mathrm{Teo}(\mathfrak{A}_A) \cup \bigcup_{s \in S} \bigcup_{k \in \alpha_s} p_\xi(c_\xi)$ is a finitely satisfiable $L(C)$-theory because the p_ξ's are types. Since $\mathrm{card}(C_s) \leq 2^\kappa$, by the Löwenheim-Skolem-Tarski theorems [Theorem 1.13] there exists a model $\mathfrak{B}'$ with $\mathrm{card}(B) \leq 2^\kappa$ — note that the models of $\mathrm{Teo}(\mathfrak{A}_A)$ has infinitely many elements of each sort. Let $\mathfrak{B}$ be the L-reduct, hence $\mathfrak{A} \preceq \mathfrak{B}$ [Lemma 1.6] and $\mathfrak{B}$ realizes the 1-types of $\mathfrak{A}$ with at most κ parameters of each sort. □

Theorem 1.23. *Let $\kappa \geq \mathrm{card}(L)$ be an infinite cardinal and $\mathfrak{A}$ an infinite L-structure such that $\mathrm{card}(A) \leq 2^\kappa$. Then, there exists a κ^+-saturated elementary extension $\mathfrak{B}$ of $\mathfrak{A}$ such that $\mathrm{card}(B) \leq 2^\kappa$.*

Proof. By the lemma 1.22, there exists an elementary sequence $(\mathfrak{A}^\alpha)_{\alpha \in 2^\kappa}$ such that $\mathfrak{A} = \mathfrak{A}^0$, $\mathfrak{A}^{\alpha+1}$ realizes every 1-type of $\mathfrak{A}^\alpha$ with at most κ parameters of each sort, $\mathfrak{A}^\alpha = \bigcup_{\xi \in \alpha} \mathfrak{A}^\xi$ for $\alpha \in 2^\kappa$ a limit ordinal, which is an elementary extension by the Tarski's Chain Lemma [Theorem 1.8], and $\mathrm{card}(A^\alpha) \leq 2^\kappa$ for each $\alpha \in 2^\kappa$. Consider $\mathfrak{B}' = \bigcup_{\alpha \in 2^\kappa} \mathfrak{A}^\alpha$, which is an elementary extension by the Tarski's Chain Lemma, and $\mathfrak{B}$ its L-reduct. Then, $\mathfrak{A} \preceq \mathfrak{B}$ and $\mathrm{card}(B) \leq 2^\kappa$. I claim that $\mathfrak{B}$ is κ^+-saturated. Indeed, let D be a sorted subset with $\mathrm{card}(D) \leq \kappa$. Then, for every $s \in S$, we have that $D_s \subseteq \bigcup_{\alpha \in 2^\kappa} A_s^\alpha$ with $\mathrm{card}(D_s) \leq \kappa$ and $\mathrm{card}(A_s^\alpha) \leq 2^\kappa$, so $D_s \subseteq A_s^{\alpha_s}$ for some $\alpha_s \in 2^\kappa$ since $\mathrm{cf}(2^\kappa) > \kappa$. Also, $\mathrm{card}(S) \leq \kappa < \mathrm{cf}(2^\kappa)$ so there is a global $\alpha \in 2^\kappa$ such that D is a sorted subset of $\mathfrak{A}^\alpha$. Then, any type with parameters D is realized in $\mathfrak{A}^{\alpha+1}$, and in $\mathfrak{B}$. □

Let T be an L-theory. A formula $\varphi(\overline{x}) \in \mathrm{For}_{\overline{x}}(L)$ *isolates* a type $\Sigma(\overline{x}) \subseteq \mathrm{For}_{\overline{x}} L$ in T if $\varphi(\overline{x}) \cup T$ is satisfiable and for any $\varphi'(\overline{x}) \in \Sigma(\overline{x})$

$$T \models \forall x_1 \dots \forall x_n \left(\varphi(x_1, \dots, x_n) \to \varphi'(x_1, \dots, x_n)\right).$$

Note that an isolated complete type is an isolated point of the space of types in the Stone's topology. Also, note that isolated types are always realized.

Theorem 1.24. (Omitting types) *Let L be a countable S-language, T a satisfiable L-theory and $\Sigma^k(\overline{x}^k) \subseteq \mathrm{For}_{\overline{x}^k} L$ non-isolated types of T for each $k \in \omega$. Then, there is a model $\mathfrak{A} \models T$ such that $\mathrm{card}(A) = \aleph_0$ and, for every $k \in \omega$, $\mathfrak{A}$ does not satisfy $\Sigma^k(\overline{x}^k)$.*

Lemma 1.25. *Let $\kappa > \mathrm{card}(L)$ be an infinite cardinal, $\mathfrak{A}$ and $\mathfrak{B}$ two equivalent L-structures such that $\mathrm{card}(A) \leq \kappa$ and $\mathfrak{B}$ is κ-saturated. Then, there is an elementary embedding $f : \mathfrak{A} \to \mathfrak{B}$.*

Proof. Let $(a_\xi)_{\xi \in \lambda}$ be an enumeration of the disjoint union of $\{A_s\}_{s \in S}$, where $\lambda \leq \kappa$ — note that $\mathrm{card}(S) \leq \mathrm{card}(L) \leq \kappa$. We define by recursion a sequence $(b_\xi)_{\xi \in \lambda}$ such that

$$\mathrm{tp}^{\mathfrak{B}}(b_\xi / \{b_\eta : \eta \in \xi\}) = \mathrm{tp}^{\mathfrak{A}}(a_\xi / \{a_\eta : \eta \in \xi\})$$

and $f_\xi : a_\eta \mapsto b_\eta$ for $\eta \leq \xi$ is an elementary map — note that we are not indicating the sorts, but every a_η has a sort, so f_ξ is a sorted function and

$\{a_\xi\}_{\xi\in\lambda}$ and $\{b_\xi\}_{\xi\in\lambda}$ are sorted sets. Since $\mathfrak{A} \equiv \mathfrak{B}$ and $\mathfrak{B}$ is κ-saturated, there exists b_0 such that $\mathrm{tp}^{\mathfrak{B}}(b_0) = \mathrm{tp}^{\mathfrak{A}}(a_0)$ and f_0 is an elementary map. The limit case is clear. Assume that for $\xi \in \lambda$ we know that f_ξ is an elementary map and that $\mathrm{tp}^{\mathfrak{B}}(b_\eta/\{b_\nu : \nu \in \eta\}) = \mathrm{tp}^{\mathfrak{A}}(a_\eta/\{a_\nu : \nu \in \eta\})$ for every $\eta \leq \xi$. Since f_ξ is an elementary map, $\mathrm{tp}^{\mathfrak{A}}(a_{\xi+1}/\{a_\eta : \eta \leq \xi\}) \in \mathbf{S}^{\mathfrak{A}}(\{a^\eta : \eta \leq \xi\}) = \mathbf{S}^{\mathfrak{B}}(\{b_\eta : \eta \leq \xi\})$. Since $\mathrm{card}(\{b_\eta \text{ of sort } s\}_{\eta\in\xi}) < \kappa$ for each sort and $\mathfrak{B}$ is κ-saturated, there exists $b_{\xi+1}$ such that

$$\mathrm{tp}^{\mathfrak{B}}(b_{\xi+1}/b_\eta : \eta \leq \xi) = \mathrm{tp}^{\mathfrak{A}}(a_{\xi+1}/a_\eta : \eta \leq \xi).$$

It is clear that $f_{\xi+1} : a_\eta \mapsto b_\eta$ for $\eta \leq \xi+1$ is $f_{\xi+1} = f_\xi \cup \{(a_{\xi+1}, b_{\xi+1})\}$, so it is an elementary map. Hence, $f : a_\xi \mapsto b_\xi$ for $\xi \in \lambda$ is an elementary embedding from $\mathfrak{A}$ to $\mathfrak{B}$. □

Lemma 1.26. *Two equivalent saturated L-structures are isomorphic.*

Proof. Let $\kappa \geq \mathrm{card}(L)$. Let $\mathfrak{A} \equiv \mathfrak{B}$ be two κ-saturated L-structures with $\mathrm{card}(A) = \mathrm{card}(B) = \kappa$. As in the lemma 1.25, let $\{a_\xi\}_{\xi\in\kappa}$ and $\{b_\xi\}_{\xi\in\kappa}$ be enumerations of the disjoint unions of the A_s's and B_s's for $s \in S$. Define by recursion a sequence $(f_\xi)_{\xi\in\kappa}$ of elementary maps such that $a_\xi \in \mathrm{Dom}\, f_\xi$ and $b_\xi \in \mathrm{Im}\, f_\xi$. Hence, it is clear that $f = \bigcup f_\xi$ is an isomorphism. □

Theorem 1.27. ***(Ryll-Nardzewski)*** *Let L be a countable S-language and T a complete L-theory whose models are infinite. Then, T is $\aleph_0$-categorical if and only if $\mathbf{S}_{\bar{s}}(T)$ is finite for any $\bar{s} \in \bigcup({}^nS)$.*

Proof. ($\Rightarrow$) Let $\bar{s}$ be such that $\mathbf{S}_{\bar{s}}(T)$ is infinite. Since $\mathbf{S}_{\bar{s}}(T)$ is a compact space [Proposition 1.18], there is a non isolated point. Let $p(\bar{x}) \in \mathbf{S}_{\bar{s}}(T)$ be non-isolated. By the Omitting Types Theorem [Theorem 1.24], there exists a model $\mathfrak{A}$ of T which does not realize p and such that $\mathrm{card}(A) = \aleph_0$. On the other hand, by the Compactness Theorem [Theorem 1.11] together with the Löwenheim-Skolem-Tarski Theorems [Theorem 1.13], $T \cup p(\bar{c})$ has a model $\mathfrak{B}'$ such that $\mathrm{card}(B) = \aleph_0$. Of course, $\mathfrak{A}$ and the L-reduct $\mathfrak{B}$ are not isomorphic, so T is not $\aleph_0$-categorical.
($\Leftarrow$) Let $\mathfrak{A}$ be a model of T with $\mathrm{card}(A) = \aleph_0$. If $\mathbf{S}_{\bar{s}}(T)$ is finite, $\mathbf{S}^{\mathfrak{A}}_s(B)$ is finite for any finite sorted subset B and any $s \in S$. Thus, every 1-type of $\mathfrak{A}$ with a finite number of parameters is isolated since the space of types is a Hausdorff's space [Proposition 1.18]. Since $\mathfrak{A}$ realizes every isolated type, $\mathfrak{A}$ is $\aleph_0$-saturated. By the lemma 1.26 we conclude that T is $\aleph_0$-categorical. □

Lemma 1.28. *Let $\kappa \geq \mathrm{card}(L)$ be an infinite cardinal, $\mathfrak{A}$ a homogeneous L-structure, B a sorted subset with $\mathrm{card}(B) < \mathrm{card}(A)$ and $\bar{c}, \bar{d}$ tuples from $\mathfrak{A}$. Then, $\mathrm{tp}(\bar{c}/B) = \mathrm{tp}(\bar{d}/B)$ if and only if there is $f \in \mathrm{Aut}(\mathfrak{A}/B)$ such that $f(\bar{c}) = \bar{d}$.*

Proof. The "if" part is clear, let us prove the "only if" part. Let $\{a_\xi\}_{\xi\in\kappa}$ be an enumeration of the disjoint union of $\{A_s\}_{s\in S}$. Note that $\kappa = \mathrm{card}(A)$. Define by recursion a sequence $(f_\xi)_{\xi\in\kappa}$ of elementary maps satisfying, for each $\eta \in \xi \in \kappa$, that $f_\eta \subseteq f_\xi$, $f_0 : B \cup \{c_1, \ldots, c_n\} \to B \cup \{d_1, \ldots, d_n\}$ with

$f_{0|B} = \mathrm{id}_B$ and $f_0(\bar{c}) = \bar{d}$, $\{a_\eta : \eta \in \xi\} \subseteq \mathrm{Dom}\ f_\xi$, $\{a_\eta : \eta \in \xi\} \subseteq \mathrm{Im}\ f_\xi$ and $\max\{\mathrm{card}(\mathrm{Dom}\ f_\xi), \mathrm{card}(\mathrm{Im}\ f_\xi)\} < \kappa$. Indeed, given an f_ξ, by the homogeneous property, since Dom f_ξ has less than κ elements for each sort, there is an elementary map extension $\widetilde{f_\xi}$ with $a_{\xi+1} \in \mathrm{Dom}\ \widetilde{f_\xi}$. Now, consider $\widetilde{f_\xi}^{-1}$, since Im $\widetilde{f_\xi}$ has less than κ elements of each sort, there is an elementary map extension $f_{\xi+1}^{-1}$ with $b_{\xi+1} \in \mathrm{Im}\ f_{\xi+1}$. Thus, $(f_\xi)_{\xi\in\kappa}$ is well defined. Hence, $f = \bigcup f_\xi$ is an automorphism such that $f(\bar{c}) = \bar{d}$ and $f \in \mathrm{Aut}(\mathfrak{A}/B)$. □

Theorem 1.29. ***(Parameters of definable sets)****.- Let $\kappa \geq \mathrm{card}(L)$ be an infinite cardinal, $\mathfrak{A}$a saturated L-structure such that $\mathrm{card}(A) = \kappa$, D a definable set and $B \subseteq A$ a sorted subset with less than κ elements of each sort. Then, D is B-definable if and only if every $f \in \mathrm{Aut}(\mathfrak{A}/B)$ leaves D invariant.*

Proof. The "only if" part is clear, let us prove the "if" part. Let $\varphi \in \mathrm{For}\, L(C)$ define D where C is a finite sorted set of parameters and $B' = B \cup C$. Then, D is B'-definable, $B \subseteq B'$ and B' has less than κ elements of each sort. Let $r : \mathbf{S}^{\mathfrak{A}}_{\bar{x}}(B') \to \mathbf{S}^{\mathfrak{A}}_{\bar{x}}(B)$ defined as $r(p) = p \cap \mathrm{For}_{\bar{x}} L(B)$. It suffices to show that $r(\langle\varphi\rangle) = \{\mathrm{tp}(\bar{c}/B) : \bar{c} \in D\}$ is closed and open. Since r is a closed map [Proposition 1.19] and $\langle\varphi\rangle$ is closed and open, it suffices to prove that $r(\langle\varphi\rangle^c) = r(\langle\varphi\rangle)^c$. Indeed, let $q \in r(\langle\varphi\rangle) \cap r(\langle\neg\varphi\rangle)$, then $q = p \cap \mathrm{For}_{\bar{x}} L(B) = p' \cap \mathrm{For}_{\bar{x}} L(B)$ with $\varphi \in p$ and $\neg\varphi \in p'$. Since $p, p' \in \mathbf{S}^{\mathfrak{A}}_{\bar{x}}(B')$ and $\mathfrak{A}$ is κ-saturated, there are $\bar{c}, \bar{d}$ such that $\bar{c} \in D$ and $\bar{d} \notin D$ and $\mathrm{tp}(\bar{c}/B) = \mathrm{tp}(\bar{d}/B) = q$. By lemma 1.28, exists an automorphism $f \in \mathrm{Aut}(\mathfrak{A}/B)$ such that $f(\bar{c}) = \bar{d}$, a contradiction since f leaves D invariant. □

1.4 Monster models

We usually need saturated structures but they do not always exist. To solve this problem we define monster models, which are an "asymptotic" way to obtain saturated "structures".

Let $\mathfrak{A}$ be an infinite L-structure. A *monster extension* $\mathfrak{C}$ of $\mathfrak{A}$ is a long sequence $(\mathfrak{A}_\alpha)_{\alpha\in\mathbb{O}\mathrm{n}}$ of L-structures such that

1. $\mathfrak{A} = \mathfrak{A}_0$;
2. $\mathfrak{A}_\beta \prec \mathfrak{A}_\alpha$ for any $\beta \in \alpha \in \mathbb{O}\mathrm{n}$;
3. and $\mathfrak{A}_{\alpha+1}$ is $|A_\alpha|^+$-saturated for any $\alpha \in \mathbb{O}\mathrm{n}$.

Let T be a complete L-theory whose models are infinite. A *monster model* $\mathfrak{C}$ of T is a monster extension of a model of T. The universe of $\mathfrak{C}$ is the family of classes $\{\mathbf{C}_s\}_{s\in S}$ defined as $\mathbf{C}_s = \bigcup_{\alpha\in\mathbb{O}\mathrm{n}} A_{\alpha s}$. A *sorted subclass* of $\mathfrak{C}$ is a family of subclasses $\{\mathbf{B}_s\}_{s\in S}$ for each $s \in S$. In particular, a sorted subset B is a sorted subclass such that, for each $s \in S$, B_s is a set. Note that a sorted subset of $\mathfrak{C}$ is a sorted subset of $\mathfrak{A}_\alpha$ for some $\alpha \in \mathbb{O}\mathrm{n}$. A *sorted function-class* $\mathbf{f} : \mathbf{B} \to \mathbf{B}'$ is a family of function-classes $\{\mathbf{f}_s\}_{s\in S}$ for each $s \in S$. We use the notation used for L-structures to monster models but we use bold letters to mark a different between classes and sets.

An *homomorphism* between monster models $\boldsymbol{\psi} : \mathfrak{C} \to \mathfrak{C}'$ is a sorted function-class such that $\boldsymbol{\psi}_{|A_\alpha} : \mathfrak{A}_\alpha \to \operatorname{Im} \boldsymbol{\psi}_{|A_\alpha}$ is an homomorphism for every $\alpha \in \mathbb{O}\mathrm{n}$. An *isomorphism* between monster models $\boldsymbol{\psi} : \mathbf{C} \to \mathbf{C}'$ is an homomorphism with an inverse. An *automorphism* in a monster model $\mathfrak{C}$ is an isomorphism from $\mathfrak{C}$ to itself.

Let $\mathfrak{C} = (\mathfrak{A}_\alpha)_{\alpha \in \mathbb{O}\mathrm{n}}$ be a monster model of an L-theory. An evaluation ϑ in $\mathfrak{C}$ is an evaluation in some $\mathfrak{A}_\alpha$ with $\alpha \in \mathbb{O}\mathrm{n}$. Thus, if $t \in \operatorname{Ter} L$ and $\varphi \in \operatorname{For} L$, $t^{\mathfrak{C}}[\vartheta] = t^{\mathfrak{A}_\alpha}[\vartheta]$ and $\mathfrak{C} \models \varphi[\vartheta] \Leftrightarrow \mathfrak{A}_\alpha \models \varphi[\vartheta]$.

Let B be a sorted subset of $\mathfrak{C}$ and $\varphi \in \operatorname{For}_{\bar{x}} L(B)$. The *$B$-definable class* defined by φ is

$$\varphi\,[\mathfrak{C}/\overline{x}] = \{\overline{a} \in \mathbf{C}_{s_1} \times \cdots \times \mathbf{C}_{s_n} \;:\; \mathfrak{C} \models \varphi[\overline{a}/\overline{x}]\} = \bigcup_{\alpha > \alpha_0} \varphi[\mathfrak{A}_\alpha].$$

where B is a sorted subset of $\mathfrak{A}_{\alpha_0}$.

We define

$$\mathbf{S}^{\mathfrak{C}}_{\bar{x}}(\mathbf{B}) = \bigcup_{\alpha \in \mathbb{O}\mathrm{n}} \mathbf{S}^{\mathfrak{A}_\alpha}_{\bar{x}}(\mathbf{B} \cap A_\alpha).$$

Note that $\mathfrak{C}$ is "set"-saturated, i.e., every type with a sorted set of parameters is realized in $\mathfrak{C}$. In particular, definable classes are finite or proper classes.

Remark. The Zermelo-Frainkel set theory with choice, ZFC, is too naive to define a monster model. Indeed, a careful reading of the proofs 1.11 and 1.23 shows that we are using the axiom of choice two define sequences of saturated structures. However, in the case of the monster model, since a long sequence is a proper class, we can not use the axiom of choice. To solve this technical problem, we work with the Bernay-Gödel set theory with global choice, BGC (The axioms of ZFC and BGC are given in the appendix A). Global choice solves this problem and ensures that monster models exists. Moreover, if we choose a well order on every $\mathfrak{A}_\alpha$ by using the axiom of global choice, we will obtain a well order (class relation) on the whole monster model $\mathfrak{C} = (\mathfrak{A}_\alpha)_{\alpha \in \mathbb{O}\mathrm{n}}$. Therefore, since $\mathfrak{C}$ is a saturated model, the results studied about saturated structures can be applied. In particular, the lemma 1.26 means that there exists just one, up to isomorphism, monster model, so we say *the* monster model. The lemma 1.25 implies that any model of a complete theory whose models have infinitely many elements of each sort is embedded into its monster model. Also, we can apply the theorem 1.29 in monster models for definable classes.

1.5 Imaginaries

On the whole, a definable set D is definable with a finite tuple of parameters $\overline{c}$, so the parameters $\overline{c}$ determine D, but D may be defined with other parameters with not relation with $\overline{c}$. For example, D could be 0-definable. We want a tuple of parameters that represents appropriately the definable set. By theorem 1.29 we can define this special type of parameters as follows:

Definition 1.30. **Canonical parameters**.- Let $\mathfrak{A}$ be a saturated L-structure, D a definable set and p a global type. A *canonical parameter* ($\mathrm{cb}(D)$) of D is a finite tuple $\overline{c}$ such that an automorphism of $\mathfrak{A}$ leaves D invariant if it fixes $\overline{c}$. Let p be a global type. A *canonical base* ($\mathrm{cb}(p)$) of p is a sorted set B with $\mathrm{card}(B) < \mathrm{card}(A)$ such that an automorphism of $\mathfrak{A}$ leaves p invariant if it fixes B. We have analogous definitions for monster models.

It is clear that there is at most one tuple of canonical parameters (or canonical bases) up to interdefinability. That is a consequence of the theorem 1.29. The problem is that we can not ensure the existence of these ones. We say that a complete theory whose models are infinite has *elimination of imaginaries* if every definable class in the monster model has a canonical parameter.

Let $\mathfrak{A}$ be a saturated L-structure. A way to build a canonical base of a definable set $D = \varphi(\overline{z}, \overline{c})[\mathfrak{A}]$ is to consider the definable equivalence relation

$$\underline{E}(\overline{x}, \overline{y}) = \forall \overline{z}(\varphi(\overline{z}, \overline{x}) \leftrightarrow \varphi(\overline{z}, \overline{y})).$$

Thus, the equivalence class of $\overline{c}$ with respect to E is a canonical parameter of D. Indeed, by definition, $f([\overline{c}]_E) = [\overline{c}]_E$ if and only if $D = \varphi(\overline{z}, f(\overline{c}))[\mathfrak{A}] = f(D)$. The problem is that $[\overline{c}]_E$ is not an element of $\mathfrak{A}$. To solve this problem we add these elements to the structures, they called *imaginaries*. So, we define the S^{eq}-language L^{eq} and the L^{eq}-expansion $\mathfrak{A}^{\mathrm{eq}}$.

$$S^{\mathrm{eq}} = \{E_{\overline{s}} \ : \ \underline{E_{\overline{s}}} \in \mathrm{For}_{\overline{s},\overline{s}}L \text{ defines a equiv. rel. with infinitely many classes}\}$$
$$L^{\mathrm{eq}} = L \cup \{\pi_{(\overline{s}, E_{\overline{s}})} \text{ function s. } : \ E_{\overline{s}} \in S^{\mathrm{eq}}\}$$
$$A^{\mathrm{eq}}_{E_{\overline{s}}} = \{[\overline{c}]_{E_{\overline{s}}} \ : \ \overline{c} \in A_{s_1} \times \cdots \times A_{s_n}\}$$
$$\pi^{\mathfrak{A}^{\mathrm{eq}}}_{(\overline{s}, E_{\overline{s}})} : \ \overline{c} \mapsto [\overline{c}]_{E_{\overline{s}}}.$$

We define $\mathfrak{C}^{\mathrm{eq}} = (\mathfrak{A}^{\mathrm{eq}}_s)$ for monster models $\mathfrak{C}$. On the other hand, given an L-theory T, we define L^{eq} and T^{eq} as follows

$$S^{\mathrm{eq}} = \{\underline{E_{\overline{s}}} \in \mathrm{For}_{\overline{s},\overline{s}}L \ : \ T \models \underline{E_{\overline{s}}} \text{ defines a equiv. rel. with infinite classes}\}$$
$$L^{\mathrm{eq}} = L \cup \{\pi_{(\overline{s}, \underline{E_{\overline{s}}})} \text{ function s. } : \ \underline{E_{\overline{s}}} \in S^{\mathrm{eq}}\}$$
$$T^{\mathrm{eq}} = T \cup \left\{\forall \widetilde{y} \exists \overline{y}(\pi_{\underline{E}}(\overline{y}) = \widetilde{y}), \forall \overline{y}_1 \forall \overline{y}_2(\pi_{\underline{E}}(\overline{y}_1) = \pi_{\underline{E}}(\overline{y}_2) \leftrightarrow \underline{E}(\overline{y}_1, \overline{y}_2))\right\}_{\underline{E} \in S^{\mathrm{eq}}}.$$

Remark. The sorts of S corresponds to the equality relation on this sort, so $S \subseteq S^{\mathrm{eq}}$. Moreover, every tuple $\overline{a} \in A_{s_1} \times \cdots \times A_{s_n}$ is associated to the imaginary $[\overline{a}]_=$ where $=$ is the equality relation between tuples. Hence, with imaginaries, there are not significant differences between tuples and elements

Lemma 1.31. *Let $\kappa \geq \mathrm{card}(L)$ be an infinite cardinal and T a κ-categorical L-theory. Then, T^{eq} is κ-categorical.*

Proof. It is clear since $\mathfrak{A} \models T \Leftrightarrow \mathfrak{A}^{\mathrm{eq}} \models T^{\mathrm{eq}}$ and any isomorphism between two L-structures $\mathfrak{A}$ and $\mathfrak{B}$ extends to an isomorphism between $\mathfrak{A}^{\mathrm{eq}}$ and $\mathfrak{B}^{\mathrm{eq}}$. $\square$

Theorem 1.32. *Let T be an L-theory, $\mathfrak{A}$ a model of T and $\varphi(\overline{x}, \widetilde{y}_1, \ldots, \widetilde{y}_N) \in \operatorname{For} L^{\mathrm{eq}}$ be such that $\overline{x}$ are of sorts in S and $\widetilde{y}_i$ is of sort $E_i \in S^{\mathrm{eq}}$ for each $i \in \{1, \ldots, N\}$. Then, there is $\varphi^*(\overline{x}, \overline{y}_1, \ldots, \overline{y}_N) \in \operatorname{For} L$ such that, for any $\overline{a}, \overline{b}_1, \ldots, \overline{b}_N$,*

$$\mathfrak{A}^{\mathrm{eq}} \models \varphi\left[\overline{a}, \pi_{E_1}^{\mathfrak{A}^{\mathrm{eq}}}(\overline{b}_1), \ldots, \pi_{E_N}^{\mathfrak{A}^{\mathrm{eq}}}(\overline{b}_N)\right] \Leftrightarrow \mathfrak{A} \models \varphi^*[\overline{a}, \overline{b}_1, \ldots, \overline{b}_N].$$

Proof. For formulas φ of L we know that $\mathfrak{A}^{\mathrm{eq}} \models \varphi[\vartheta]$ if and only if $\mathfrak{A} \models \varphi[\vartheta]$ — note that satisfaction is independent from evaluations of variables of sorts from S^{eq}, so ϑ may be an evaluation in $\mathfrak{A}$. First, note that a term of L^{eq} is either a term of L, or $\pi_E(t_1, \ldots, t_n)$ for some $t_1, \ldots, t_n \in \operatorname{Ter} L$, or a variable of sort in S^{eq}. We prove the theorem by induction over the complexity of φ.

If $\varphi = R(t_1, \ldots, t_n)$, since R is in L, $\varphi \in \operatorname{For} L$, so $\varphi^* = \varphi$. If $\varphi = t \doteq t'$ there are three cases. If $t, t' \in \operatorname{Ter} L$, $\varphi \in \operatorname{For} L$, so $\varphi^* = \varphi$. If t (or t') is a variable $\widetilde{y}$ of sort $E \in S^{\mathrm{eq}}$, we know that $\mathfrak{A}^{\mathrm{eq}} \models \widetilde{y} \doteq t'[\vartheta, \pi_E^{\mathfrak{A}^{\mathrm{eq}}}(\overline{b})/\widetilde{y}]$ if and only if $\mathfrak{A}^{\mathrm{eq}} \models \pi_E(\overline{y}) \doteq t'[\vartheta, \overline{b}/\overline{y}]$. So, we reduce the problem to the following case. If $t = \pi_E(t_1, \ldots, t_n)$ and $t' = \pi_E(t'_1, \ldots, t'_n)$ we have that $\mathfrak{A}^{\mathrm{eq}} \models t \doteq t'[\vartheta]$ if and only if $\mathfrak{A}^{\mathrm{eq}} \models \underline{E}(t_1, \ldots, t_n, t'_1, \ldots, t'_n)[\vartheta]$, where $\varphi^* = \underline{E}(t_1, \ldots, t_n, t'_1, \ldots, t'_n) \in \operatorname{For} L$.

If $\varphi = \neg\varphi_1$ or $\varphi = \varphi_1 \vee \varphi_2$, we conclude that $\varphi^* = \neg\varphi_1^*$ and $\varphi^* = \varphi_1^* \vee \varphi_2^*$ by induction hypothesis. We have to prove the case $\varphi = \exists\widetilde{y}\varphi_1$. There are two cases. If $\widetilde{y}$ is of sort in S, by induction hypothesis, $\varphi^* = \exists\widetilde{y}\varphi_1^*$. Assume $\widetilde{y}$ is of sort $E \in S^{\mathrm{eq}}$. Let $\vartheta^{\mathrm{eq}} = \overline{a}/\overline{x}, \pi_{E_1}^{\mathfrak{A}^{\mathrm{eq}}}(\overline{b}_1)/\widetilde{y}_1, \ldots, \pi_{E_N}^{\mathfrak{A}^{\mathrm{eq}}}(\overline{b}_N)/\widetilde{y}_N$ and $\vartheta = \overline{a}/\overline{x}, \overline{b}_1/\overline{y}_1, \ldots, \overline{b}_N/\overline{y}_N$. We know that $\mathfrak{A}^{\mathrm{eq}} \models \varphi[\vartheta^{\mathrm{eq}}]$ if and only if there is $c \in A_E^{\mathrm{eq}}$ such that $\mathfrak{A}^{\mathrm{eq}} \models \varphi_1[\vartheta^{\mathrm{eq}}, c/\widetilde{y}]$. Since $A_E^{\mathrm{eq}} = \{\pi_E^{\mathfrak{A}^{\mathrm{eq}}}(\overline{b}) \ : \ \overline{b} \text{ from } A\}$, there are $\overline{b}$ such that $c = \pi_E^{\mathfrak{A}^{\mathrm{eq}}}(\overline{b})$. Thus, we can apply the induction hypothesis to get $\varphi_1^* \in \operatorname{For} L$ such that $\mathfrak{A}^{\mathrm{eq}} \models \varphi[\vartheta^{\mathrm{eq}}]$ if and only if there is $\overline{b}$ such that $\mathfrak{A} \models \varphi_1^*(\overline{y})[\vartheta, \overline{b}]$. Hence, $\varphi^* = \exists\overline{y}\varphi_1^*$ is such that $\mathfrak{A}^{\mathrm{eq}} \models \varphi[\vartheta^{\mathrm{eq}}]$ if and only if $\mathfrak{A} \models \varphi^*[\vartheta]$. $\square$

Remark. The last theorem 1.32 implies that a definable set of $\mathfrak{M}^{\mathrm{eq}}$ included in $\mathfrak{M}$ is definable in $\mathfrak{M}$. Therefore, a complete type p in $\mathfrak{M}$ has a unique extension p^{eq} to a complete type in $\mathfrak{M}^{\mathrm{eq}}$ with the same parameters. Then, we define $\mathrm{cb}(p) := \mathrm{cb}(p^{\mathrm{eq}})$ and the comments just made imply it is well defined.

Corollary 1.33. *Let T be a complete L-theory whose models are infinite. Then, T^{eq} has elimination of imaginaries.*

Proof. Let $\mathbf{D}$ be a definable class, then $\mathbf{D} = \varphi(\overline{u}, \overline{c})[\mathfrak{C}^{\mathrm{eq}}]$. By the substitution lemma 1.3, assume that $\overline{c}$ is from $\mathbf{C}$. Now, by the theorem 1.32, there is $\varphi^* \in \operatorname{For} L$ such that

$$\mathbf{D} = \left\{\left(\overline{a}, \pi_{E_1}^{\mathfrak{C}^{\mathrm{eq}}}(\overline{b}_1), \ldots, \pi_{E_N}^{\mathfrak{C}^{\mathrm{eq}}}(\overline{b}_N)\right) \ : \ \mathfrak{C} \models \varphi^*(\overline{x}, \overline{y}_1, \ldots, \overline{y}_N, \overline{c})[\overline{a}, \overline{b}_1, \ldots, \overline{b}_N]\right\}.$$

Let

$$\underline{E}(\overline{z}, \overline{w}) = \forall\overline{x}, \overline{y}\left(\varphi^*(\overline{x}, \overline{y}, \overline{z}) \leftrightarrow \left(\exists\overline{y}'\varphi^*(\overline{x}, \overline{y}', \overline{w}) \wedge \bigwedge_i \underline{E}_i(\overline{y}_i, \overline{y}'_i)\right)\right).$$

$\underline{E} \in \mathrm{For}\, L$ and defines the equivalence relation $E(\bar{c},\bar{d}) \Leftrightarrow \varphi(\mathfrak{C}^{\mathrm{eq}},\bar{c}) = \varphi(\mathfrak{C}^{\mathrm{eq}},\bar{d})$. If E has infinite equivalence classes, $\mathrm{cb}(\mathbf{D}) = \pi_E^{\mathfrak{C}^{\mathrm{eq}}}(\bar{c})$. If not, choose a tuple $\bar{a}$ of elements of each class of E, then $\bar{a} = \mathrm{cb}(\mathbf{D})$. $\square$

Corollary 1.34. *Let $\kappa \geq \mathrm{card}(L)$ be an infinite cardinal and $\mathfrak{A}$ an L-structure. Then, $\mathfrak{A}$ is κ-saturated if and only if $\mathfrak{A}^{\mathrm{eq}}$ is κ-saturated. In particular, $\mathfrak{A}$ is saturated if and only if $\mathfrak{A}^{\mathrm{eq}}$ is saturated.*

Proof. Since $\mathfrak{A}^{\mathrm{eq}}$ is an expansion of $\mathfrak{A}$, the "if" part is clear. We prove the "only if" part. Firstly, note that, if $\mathrm{card}(A) \geq \kappa$, every 0-definable equivalence relation with infinite classes $E \in S^{\mathrm{eq}}$ has at least κ classes. Thus, $\mathrm{card}(A_s^{\mathrm{eq}}) \geq \kappa$ for each $s \in S \cup S^{\mathrm{eq}}$. Hence, $\mathrm{card}(A) = \kappa$ implies $\mathrm{card}(A^{\mathrm{eq}}) = \kappa$. Now, let B be a sorted subset of $\mathfrak{A}^{\mathrm{eq}}$ such that $\mathrm{card}(B) < \kappa$, and let B^* be a sorted subset of $\mathfrak{A}$ such that $\mathrm{card}(B^*) < \kappa$ and every imaginary element of B is the equivalence class of a tuple from B^*. Let $p \in \mathbf{S}_x^{\mathfrak{A}^{\mathrm{eq}}}(B)$ with x of sort $E \in \mathrm{For}_{\bar{s},\bar{s}} L$, and consider

$$p^* = \{\varphi^*(\bar{y},\bar{b}^*) \,:\, \varphi(x,\bar{b}) \in p \text{ where } \bar{b} \text{ are the classes of } \bar{b}^* \text{ from } B^*\}.$$

Thus, by the theorem 1.32, p^* is a type. Since $\mathfrak{A}$ is κ-saturated, there is tuple $\bar{a}$ realizing p^*. Hence, by the theorem 1.32, $[\bar{a}]_E$ realizes p. $\square$

Lemma 1.35. *Let T be a complete L-theory whose models are infinite and with at least a 0-definable element of each sort and two 0-definable elements of one sort. Then, T has elimination of imaginaries if and only if, for every 0-definable class $\mathbf{D}$ of $\mathfrak{C}^{\mathrm{eq}}$, there exists a 0-definable one-to-one function class $\mathbf{f}$ from $\mathbf{D}$ to $\mathbf{C}^m$ for some m.*

Proof. ($\Leftarrow$) Given a definable class $\mathbf{D}$ of $\mathfrak{C}$, let $\mathrm{cb}(\mathbf{D}) = c \in \mathbf{C}_E^{\mathrm{eq}}$, and let $\mathbf{f}$ be a 0-definable one-to-one function class from $\mathbf{C}_E^{\mathrm{eq}}$ to $\mathfrak{C}$. Thus, $\mathbf{f}(c)$ and c are interdefinable, so $\mathbf{f}(c)$ is a canonical base of $\mathbf{D}$ in $\mathfrak{C}$. Therefore, T has elimination of imaginaries.
($\Rightarrow$) Let $\mathbf{D} \subseteq \mathbf{C}_{E_1}^{\mathrm{eq}} \times \cdots \times \mathbf{C}_{E_l}^{\mathrm{eq}}$ be 0-definable. We want to find an $\bar{s} \in S^N$ and a 0-definable one-to-one function class $\mathbf{f} :\, \mathbf{D} \to \mathbf{C}_{s^1} \times \cdots \times \mathbf{C}_{s^N}$. Since T has elimination of imaginaries, every element of $\mathfrak{C}^{\mathrm{eq}}$ is 0-interdefinable with a tuple of elements of $\mathfrak{C}$. Indeed, given $[c] \in \mathbf{C}_E^{\mathrm{eq}}$, let $\bar{a}$ from $\mathfrak{C}$ be the canonical base of $E(\mathfrak{C},c)$, then $[c]$ and $\bar{a}$ are 0-interdefinable. Therefore, for each $i \leq l$, for each $c \in \mathbf{C}_{E_i}^{\mathrm{eq}}$, there is an $\bar{a}_c$ from $\mathfrak{C}$ of sorts $\bar{s}_c \in S^{n_c}$ such that c and $\bar{a}_c$ are 0-interdefinable. Let $\underline{\mathbf{f}_c}(x,\bar{y}) \in \mathrm{For}_{E_i,\bar{s}} L^{\mathrm{eq}}$ be the formula which states the interdefinitions of c and $\bar{a}_c$. Let $\mathbf{D}_c \subseteq \mathbf{C}_{E_i}^{\mathrm{eq}}$ be the maximal 0-definable class such that $\underline{\mathbf{f}}_c$ is a 0-definable one-to-one function from $\mathbf{D}_c$ to $\mathfrak{C}$. Now, for each $\bar{c} = (c^1,\ldots,c^l) \in \mathbf{D}$, let $\mathbf{D}_{\bar{c}} = (\mathbf{D}_{c^1} \times \cdots \times \mathbf{D}_{c^l}) \cap \mathbf{D}$. Since $\bar{c} \in \mathbf{D}_{\bar{c}}$, it is clear that $\mathbf{D} = \bigcup_{\bar{c}\in\mathbf{D}} \mathbf{D}_{\bar{c}}$. Thus, $\langle\underline{\mathbf{D}}\rangle = \bigcup_{\bar{c}\in\mathbf{D}} \langle\underline{\mathbf{D}_{\bar{c}}}\rangle$. Since $\mathbf{S}_{E_1,\ldots,E_l}^{\mathfrak{C}^{\mathrm{eq}}}$ is a Hausdorff's compact space and $\langle\underline{\mathbf{D}}\rangle$ is closed, there exists a finite list $\bar{c}_1,\ldots,\bar{c}_N$ such that $\mathbf{D} = \mathbf{D}_{\bar{c}_1} \cup \cdots \cup \mathbf{D}_{\bar{c}_N}$. Let $\psi_s \in \mathrm{For}_s L^{\mathrm{eq}}$ define an element b_s, for each $s \in S$, and ψ_{s_0}, ψ'_{s_0} define two elements b_{s_0} and b'_{s_0}. Define $\mathbf{f}$ as follows

$$\underline{\mathbf{f}}(\bar{x},\bar{y}^{1,1},\ldots,\bar{y}^{l,N},\bar{z}) = \bigvee_{i=1}^{N} \left(\left(\underline{\mathbf{D}_{\bar{c}_i}}(\bar{x}) \wedge \bigwedge_{j<i} \neg\underline{\mathbf{D}_{\bar{c}_j}}(\bar{x}) \right) \right.$$

$$\rightarrow \left(\bigwedge_{t \leq l} \underline{\mathbf{f}_{c_i^t}}(x_t, \overline{y}^{t,i}) \wedge \bigwedge_{j \neq i} \bigwedge_{t \leq l} \bigwedge_{k \leq n_{c_j^t}} \psi_{s^k_{\bar{c}_j}}(y_k^{t,j}) \wedge \bigwedge_{w \leq i} \psi_{s_0}(z_w) \wedge \bigwedge_{i < w \leq N} \psi'_{s_0}(z_w) \right) \Bigg).$$

Where $\overline{y}^{t,i}$ are of sorts $\overline{s}_{c_i^t}$, $\overline{z}$ are of sorts s_0 and $\overline{x}$ are of sorts $E_1, \ldots, E_l$. It is clear that $\mathbf{f}$ is a 0-definable one-to-one function class from $\mathbf{D}$ to $\mathfrak{C}$. Indeed, $\{\mathbf{D}_{\bar{c}_i} \setminus \bigcup_{j<i} \mathbf{D}_{\bar{c}_j}\}_{i \leq N}$ is a partition and, for $\overline{d} \in \mathbf{D}_{\bar{c}_i} \setminus \bigcup_{j<i} \mathbf{D}_{\bar{c}_j}$, we have that

$$\mathbf{f}(\overline{d}) = (b_{\overline{s}_{c_1^1}}, \ldots, b_{\overline{s}_{c_1^l}}, \ldots, \mathbf{f}_{c_i^1}(\overline{d}), \ldots, \mathbf{f}_{c_i^l}(\overline{d}), \ldots, \underbrace{b_{s_0}, \ldots, b_{s_0}}_{i}, \underbrace{b'_{s_0}, \ldots, b'_{s_0}}_{N-i}),$$

where $b_{\overline{s}} = (b_{s_1}, \ldots, b_{s_n})$. $\square$

Lemma 1.36. *Let $\mathfrak{A}$ be a saturated L-structure, D a definable set,* $\mathrm{cb}(D)$ *in* $\mathfrak{A}^{\mathrm{eq}}$ *and B a sorted subset of* $\mathfrak{A}^{\mathrm{eq}}$ *with* $\mathrm{card}(B) < \mathrm{card}(A^{\mathrm{eq}})$. *Then, the following are equivalent:*
(1) $\mathrm{cb}(D) \in \mathrm{acl}^{\mathrm{eq}}(B)$.
(2) D is $\mathrm{acl}^{\mathrm{eq}}(B)$-definable.
(3) D has a finite number of conjugates over B.
(4) D is a union of classes of a B-definable equivalence relation which has a finite number of classes.

Proof. Remember that $\mathfrak{A}^{\mathrm{eq}}$ is saturated by the corollary 1.34.
(1)⇔(2) We apply the theorem of parameters of definable sets [Theorem 1.29]. D is $\mathrm{acl}^{\mathrm{eq}}(B)$-definable if and only if it is left invariant by every automorphism fixing $\mathrm{acl}^{\mathrm{eq}}(B)$. Therefore, D is $\mathrm{acl}^{\mathrm{eq}}(B)$-definable if and only if $\mathrm{cb}(D)$ is left invariant by every automorphism fixing $\mathrm{acl}^{\mathrm{eq}}(B)$, if and only if $\mathrm{cb}(D) \in \mathrm{dcl}^{\mathrm{eq}}(\mathrm{acl}^{\mathrm{eq}}(B))$. Since $\mathrm{acl}^{\mathrm{eq}}(B) \subseteq \mathrm{dcl}^{\mathrm{eq}}(\mathrm{acl}^{\mathrm{eq}}(B)) \subseteq \mathrm{acl}^{\mathrm{eq}}(\mathrm{acl}^{\mathrm{eq}}(B))$, it suffices to prove that $\mathrm{acl}^{\mathrm{eq}}(\mathrm{acl}^{\mathrm{eq}}(B)) \subseteq \mathrm{acl}^{\mathrm{eq}}(B)$. That will be proved in the theorem 2.48.
(1)⇔(3) Let $f, f' \in \mathrm{Aut}(\mathfrak{A}^{\mathrm{eq}}/B)$, then $f(D) = f'(D)$ if and only if $f(\mathrm{cb}(D)) = f'(\mathrm{cb}(D))$. Thus, there are as many conjugates of D over B as many conjugates of $\mathrm{cb}(D)$ over B. Since the number of conjugates of $\mathrm{cb}(D)$ over B is finite if and only if $\mathrm{cb}(D) \in \mathrm{acl}^{\mathrm{eq}}(B)$.
(3)⇒(4) Suppose that $D_0, \ldots, D_n$ are the conjugates of D over B. Let $\varphi(\overline{x}, y) \in \mathrm{For}\, L^{\mathrm{eq}}$ be such that $\varphi(\overline{x}, \mathrm{cb}(D))[\mathfrak{A}^{\mathrm{eq}}] = D$. Then, $D_i = \varphi(\overline{x}, f_i(\mathrm{cb}(D)))$ for some $f_i \in \mathrm{Aut}(\mathfrak{A}^{\mathrm{eq}}/B)$. Consider

$$\underline{E}(\overline{x}, \overline{x}') = \bigwedge_{i \leq n} \left(\varphi(\overline{x}, f_i(\mathrm{cb}(D))) \leftrightarrow \varphi(\overline{x}, f_i(\mathrm{cb}(D)))\right).$$

It is clear that $E = \underline{E}[\mathfrak{A}^{\mathrm{eq}}]$ is a definable equivalence relation. Since every automorphism fixing B leaves it invariant, E is B-definable by the theorem of parameters of definable sets [Theorem 1.29]. Finally, the equivalence classes are $\bigcup_{i \in I} D_i \setminus (\bigcup_{i \notin I} D_i)$, so there are at most 2^n equivalence classes, that is a finite number.
(4)⇒(3) Let $D = \bigcup_{i=1}^{k} \underline{E}(\overline{x}, \overline{d}_i)[\mathfrak{A}^{\mathrm{eq}}]$ for some $\overline{d}_1, \ldots, \overline{d}_k \in D$ where E is a

B-definable equivalence relation with n equivalence classes. Thence, for any $f \in \mathrm{Aut}(\mathfrak{A}/B)$, we have that $f(D) = \bigcup_{i=1}^{n} \underline{E}(\overline{x}, f(\overline{d}_i))[\mathfrak{A}^{\mathrm{eq}}]$. There are at most $\binom{n}{k}$ unions of k equivalence classes of E, so D has at most $\binom{n}{k}$ conjugates over B. $\square$

Corollary 1.37. *Let $\mathfrak{A}$ be a saturated L-structure, a an element of $\mathfrak{A}^{\mathrm{eq}}$ and B a sorted subset of $\mathfrak{A}^{\mathrm{eq}}$ with $\mathrm{card}(B) < \mathrm{card}(A^{\mathrm{eq}})$. Then, $\mathrm{stp}(a/B)$ is axiomatized by the set of $\mathrm{acl}^{\mathrm{eq}}(B)$-formulas $\underline{\Sigma}$ defining the elements of*

$$\Sigma = \{\underline{E}(x, a)[\mathfrak{A}^{\mathrm{eq}}] \,:\, \underline{E} \text{ an equiv. rel. } B\text{-def. with a finite number of classes}\}.$$

Proof. First of all, every $\underline{E}(x, a)[\mathrm{A}^{\mathrm{eq}}] \in \Sigma$ is $\mathrm{acl}^{\mathrm{eq}}(B)$-definable by the last lemma 1.36. Let $p \in \langle\underline{\Sigma}\rangle$ and $\varphi \in p$. We want to prove that $p = \mathrm{stp}(a/B)$. Since $\varphi \in \mathrm{For}_x L^{\mathrm{eq}}(\mathrm{acl}^{\mathrm{eq}}(B))$, by the lemma 1.36 there is a B-definable equivalence relation $\underline{E}$ with a finite number of classes such that $\varphi[\mathfrak{A}^{\mathrm{eq}}] = \bigcup_{i=1}^{n} \underline{E}(x, d_i)[\mathfrak{A}^{\mathrm{eq}}]$ for some $d_1, \ldots, d_n$. Now, $\underline{E}(x, a)[\mathfrak{A}^{\mathrm{eq}}] \in \Sigma$, so $\{\varphi(x), \underline{E}(x, a)\}$ is satisfiable. Hence, $\varphi[\mathfrak{A}^{\mathrm{eq}}] = \underline{E}(x, a)[\mathfrak{A}^{\mathrm{eq}}]$. So, in particular, $\mathfrak{A}^{\mathrm{eq}} \models \varphi[a]$. $\square$

Remark. The results 1.34 and 1.37 are also true for monster models. Note that if $\mathrm{Teo}(\mathfrak{A})$ has elimination of imaginaries the use of $^{\mathrm{eq}}$ is not necessary.

A complete theory whose models are infinite *eliminates finite imaginaries* if every finite set of n-tuples of the monster model $\mathfrak{C}$ has a canonical parameter in $\mathfrak{C}$. A complete theory whose models are infinite has *weak elimination of imaginaries* if for every element c of $\mathfrak{C}^{\mathrm{eq}}$ there is a finite tuple $\overline{d}$ from $\mathfrak{C}$ such that $c \in \mathrm{dcl}^{\mathrm{eq}}(\overline{d})$ and $\overline{d}$ is from $\mathrm{acl}^{\mathrm{eq}}(c)$.

***Lemma* 1.38.** *Let T be a complete L-theory whose models are infinite. Then, T has elimination of imaginaries if and only if it has weak elimination of imaginaries and eliminates finite imaginaries.*

Proof. The "only if" part is clear. Let us prove the "if" part. Given a definable class $\mathbf{D}$, let $c = \mathrm{cb}(\mathbf{D})$ in $\mathfrak{C}^{\mathrm{eq}}$, let $\overline{d}$ from $\mathrm{acl}^{\mathrm{eq}}(c)$ be such that $c \in \mathrm{dcl}(\overline{d})$ and let $d = \mathrm{cb}(\{\overline{d}_1, \ldots, \overline{d}_n\})$ be in $\mathfrak{C}$ where $\overline{d}_1, \ldots, \overline{d}_n$ are the conjugates of $\overline{d}$ over c. I claim that c and d are interdefinable so $\mathrm{cb}(\mathbf{D}) = d$. Indeed, by corollary 1.34 and the theorem 1.29, it suffices to prove that an automorphism $\mathbf{f}$ fixes c if and only if it fixes d. If $\mathbf{f}(c) = c$, then $\mathbf{f}$ takes the conjugates of $\overline{d}$ over c to conjugates. Thus, $\mathbf{f}$ fixes d, i.e., $d \in \mathrm{dcl}^{\mathrm{eq}}(c)$. On the other hand, if $\mathbf{f}(d) = d$, $\mathbf{f}(\overline{d}) = \overline{d}_i$ for some i. Let $\mathbf{g}$ be an automorphism fixing c such that $\mathbf{g}(\overline{d}) = \overline{d}_i$. Then, $\mathbf{g}^{-1} \circ \mathbf{f}$ fixes $\overline{d}$, so it fixes c. Since $\mathbf{g}$ fixes c, $\mathbf{f}$ fixes c. Hence, $c \in \mathrm{dcl}^{\mathrm{eq}}(d)$. $\square$

Theorem 1.39. *Let T be a complete L-theory whose models are infinite. Then, T has weak elimination of imaginaries, provided $\mathrm{acl}(\emptyset)$ is infinite and every definable class $\mathbf{D} \subseteq \mathbf{C}$ of $\mathfrak{C}$ is either finite or cofinite.*

Proof. Let $[\overline{c}]_E \in \mathbf{C}_E^{\mathrm{eq}}$, it suffices to prove that $\mathbf{D} = \underline{E}(\overline{x}, \overline{c})[\mathfrak{C}]$ and $\mathrm{acl}^{\mathrm{eq}}([\overline{c}]_E)$ are not disjoint. So, it suffices to prove that in every definable class $\mathbf{D} \subseteq \mathbf{C}^n$ there is an element of $\mathrm{acl}^{\mathrm{eq}}(\mathrm{cb}(\mathbf{D}))$. We prove it by induction on n. For $n = 1$, either $\mathbf{D}$ is finite or cofinite. If $\mathbf{D}$ is finite, $\mathbf{D}$ is a subset of $\mathrm{acl}^{\mathrm{eq}}(\mathrm{cb}(\mathbf{D}))$. If

$\mathbf{D}$ is cofinite, since $\mathrm{acl}^{\mathrm{eq}}(\emptyset) \subseteq \mathrm{acl}^{\mathrm{eq}}(\mathrm{cb}(\mathbf{D}))$ is infinite, there is an element of $\mathrm{acl}^{\mathrm{eq}}(\emptyset)$ in $\mathbf{D}$. Assume the case $n-1$, consider $\underline{\mathbf{D}'} = \exists x_1 \underline{\mathbf{D}}(x_1, \ldots, x_n)[\mathfrak{C}]$. There is $a_1 \in \mathbf{D}'$ in $\mathrm{acl}^{\mathrm{eq}}(\mathrm{cb}(\mathbf{D}'))$. By the theorem 1.29, $\mathrm{cb}(\mathbf{D}') \in \mathrm{dcl}^{\mathrm{eq}}(\mathrm{cb}(\mathbf{D}))$, so $\mathrm{acl}^{\mathrm{eq}}(\mathrm{cb}(\mathbf{D}')) \subseteq \mathrm{acl}^{\mathrm{eq}}(\mathrm{cb}(\mathbf{D}'))$. Let $\underline{\mathbf{D}''} = \underline{\mathbf{D}}(a_1, x_2, \ldots, x_n)$. By induction hypothesis there are $a_2, \ldots, a_n \in \mathrm{acl}^{\mathrm{eq}}(\mathrm{cb}(\mathbf{D}''))$ such that $\overline{a} \in \mathbf{D}$. By the theorem 1.29, since $a_1 \in \mathrm{acl}^{\mathrm{eq}}(\mathrm{cb}(\mathbf{D}))$, we conclude that $\mathrm{cb}(\mathbf{D}'') \in \mathrm{acl}^{\mathrm{eq}}(\mathrm{cb}(\mathbf{D}))$. Hence, $\overline{a} \in \mathbf{D}$ is from $\mathrm{acl}^{\mathrm{eq}}(\mathrm{cb}(\mathbf{D}))$. $\square$

2 Morley's rank

In this chapter we study the fundamental concept of Morley's rank which is a general version of dimension for structures.

Firstly, we define the stable theories and characterize these ones by the order property [Theorem 2.7].

Next, in the second section, we define Morley's rank and study its most basic properties [Propositions 2.11 and 2.12]. In particular, the lemma 2.14 describes the case of definable sets without Morley's rank and implies that ω-stable theories are totally transcendental, i.e., that every definable set has Morley's rank [Theorem 2.18].

In the third section we define Morley's rank for types and study its most basic properties [Proposition 2.20 and Theorem 2.23]. The main result of this section is the definability of types with Morley's rank [Theorem 2.28 and Corollary 2.29].

A concept of dimension involves a concept of independence. We study the forking independence (the independence associated to Morley's rank) in the section four. Its basic properties are transitivity, monotonicity, finiteness and symmetry [Proposition 2.31 and Theorem 2.33]. Also, we study the relation of forking and canonical bases [Theorem 2.37] and the characterization of forking as heirs and coheirs [Theorem 2.42]

We study the pregeometries of strongly minimal definables sets (classes) in the section five. In this particular case, forking independence is the same that algebraic independence [Theorem 2.51]. Also, we study almost strongly minimal definable sets (classes) [Theorems 2.55 and 2.56]. The most significant result is the characterization of locally modular pregeometries [Theorem 2.60].

We end this chapter with the definition of orthogonality.

Notation. In the rest of this memoir and except otherwise stated, $\mathfrak{M}$ will be an $\aleph_0$-saturated L-structure and T will be a complete L-theory whose models are infinite.

2.1 Stable theories

Let $\kappa \geq \operatorname{card}(L)$ be an infinite cardinal. We say that a theory T is *κ-stable* if $\operatorname{card}\left(\mathbf{S}_{\overline{x}}^{\mathfrak{M}}(A)\right) \leq \kappa$ for every model $\mathfrak{M}$ of T, any tuple of variables $\overline{x}$ and every sorted subset A such that $\operatorname{card}(A) \leq \kappa$. A theory T is *stable* if is κ-stable for some cardinal κ.

We first prove that to study κ-stability we only need to consider types in one single variable.

Proposition 2.1. *Let $\kappa \geq \operatorname{card}(L)$ be an infinite cardinal. Then, T is κ-stable if and only if* $\operatorname{card}\left(\mathbf{S}_{x}^{\mathfrak{M}}(A)\right) \leq \kappa$ *for every $\mathfrak{M} \models T$, every sorted subset A with* $\operatorname{card}(A) \leq \kappa$ *and every single variable x.*

Proof. By induction on n. Let $\mathfrak{C}$ be the monster model of T. For $n = 1$ by hypothesis. Let $n > 1$ and $\bar{x} = (x_1, \ldots, x_n)$ variables and consider $\mathbf{S}_{\bar{x}}^{\mathfrak{C}}(A)$ with

$\mathrm{card}(A) \leq \kappa$. By induction hypothesis $\mathrm{card}\left(\mathbf{S}^{\mathfrak{C}}_{x_2 \ldots x_n}(A)\right) \leq \kappa$, so there is a set $B' = \{\bar{b}_\alpha\}_{\alpha\in\kappa}$ in $\mathfrak{C}$ of realizations of the types of $\mathbf{S}^{\mathfrak{C}}_{x_2\ldots x_n}(A)$. For every $\alpha \in \kappa$, let A_α be the result of adding $\bar{b}_\alpha$ to A, then $\mathrm{card}\left(\mathbf{S}^{\mathfrak{C}}_{x_1}(A_\alpha)\right) \leq \kappa$. So, there is a set $B_\alpha = \{b^\beta_\alpha\}_{\beta\in\kappa}$ in $\mathfrak{C}$ of realizations of the type of $\mathbf{S}^{\mathfrak{C}}_{x_1}(A_\alpha)$. Consider the set $B = \{(b^\beta_\alpha, \bar{b}_\alpha) \ : \ \alpha, \beta \in \kappa\}$. Since $\kappa^2 = \kappa$, we have that $\mathrm{card}(B) = \kappa$. For every type $p \in \mathbf{S}^{\mathfrak{C}}_{\bar{x}}(A)$, let $\bar{b}_\alpha \in B'$ be such that $\mathfrak{C} \models \exists x_1 \varphi[\bar{b}_\alpha]$ for each $\varphi \in p$, and let $b^\beta_\alpha \in B_\alpha$ be such that $\mathfrak{C} \models \varphi[b^\beta_\alpha, \bar{b}_\alpha]$ for each $\varphi \in p$. Thus, $\mathbf{S}^{\mathfrak{C}}_{\bar{x}}(A) \subseteq \{\mathrm{tp}(b^\beta_\alpha, \bar{b}_\alpha/A) \ : \ \alpha, \beta \in \kappa\}$. Therefore, $\mathrm{card}\left(\mathbf{S}^{\mathfrak{C}}_{\bar{x}}(A)\right) \leq \kappa$. $\square$

Theorem 2.2. *Let* $\mathrm{card}(L) \leq \lambda \leq \kappa$ *be cardinals such that* λ *is regular and* T *a* κ*-stable theory. Then,* T *has a* λ*-saturated model* $\mathfrak{M}$ *such that* $\mathrm{card}(M) = \kappa$*. In particular, if* κ *is regular, there is a saturated model* $\mathfrak{M}$ *of* T *such that* $\mathrm{card}(M) = \kappa$.

Proof. We adapt the proofs of 1.22 and 1.23. Define by recursion a continuous sequence $(\mathfrak{M}_\alpha)_{\alpha\in\lambda}$ of models of T such that $\mathfrak{M}_\xi \prec \mathfrak{M}_\alpha$ for $\xi \in \alpha$, every type of $\mathfrak{M}_\alpha$ is realized in $\mathfrak{M}_{\alpha+1}$ and $\mathrm{card}(\mathfrak{M}_{\alpha+1}) \leq \kappa$ — note that this is possible since $\mathrm{card}(\mathbf{S}^{\mathfrak{M}_\alpha}_x) \leq \kappa$. Then, $\mathfrak{M} = \bigcup_{\alpha\in\lambda} \mathfrak{M}_\alpha$ is a model of T such that $\mathrm{card}(M_s) \leq \kappa$. Since λ is regular, $\mathfrak{M}$ is λ-saturated. Also, if $\mathrm{card}(\mathfrak{M}_0) = \kappa$, then $\mathfrak{M}$ has cardinality κ. $\square$

Let $\mathfrak{M}$ be an L-structure, B a sorted subset and $\varphi \in \mathrm{For}_{\bar{s},\bar{s}'} L$ where $\bar{s}' = (s'_1, \ldots, s'_n)$. The *$\varphi$-code of a type p over B* is the set of tuples $V \subseteq B_{s'_1} \times \cdots \times B_{s'_n}$ such that

$$\bar{b} \in V \Leftrightarrow \varphi(\bar{x}, \bar{b}) \in p.$$

We write $\mathcal{S}^{\mathfrak{M}}_\varphi(B)$ for the set of φ-codes of types over B.

Let T be an L-theory and $\varphi(\bar{x}, \bar{y}) \in \mathrm{For}_{\bar{s},\bar{s}'} L$. We say that φ has *the order property* in T if there are a model $\mathfrak{M} \models T$ and sequences $(\bar{a}_i)_{i\in\omega}$ and $(\bar{b}_j)_{j\in\omega}$ such that

$$T \models \varphi[\bar{a}_i/\bar{x}, \bar{b}_j/\bar{y}] \Leftrightarrow i < j.$$

Let T be an L-theory and $\varphi(\bar{x}, \bar{y}) \in \mathrm{For}_{\bar{s},\bar{s}'} L$. We say that φ has *the binary tree property* in T if there are a model $\mathfrak{M} \models T$ and a sequence $(\bar{b}_i)_{i\in{}^{<\omega}2}$ such that for all $\sigma \in {}^\omega 2$ the set

$$\{\varphi^{\sigma(n)}(\bar{x}, \bar{b}_{\sigma|n}) \ : \ n \in \omega\}$$

is a type, where $\varphi^0 := \varphi$ and $\varphi^1 := \neg\varphi$.

Lemma 2.3. *Let* T *be an* L*-theory,* $\varphi(\bar{x}, \bar{y}) \in \mathrm{For}_{\bar{s},\bar{s}'} L$ *with the order property and* $(I, <_I)$ *be a linear order. Then, there are a model* $\mathfrak{M} \models T$ *and sequences* $(\bar{a}_i)_{i\in I}$ *and* $(\bar{b}_j)_{j\in I}$ *such that* $\mathfrak{M} \models \varphi[\bar{a}_i, \bar{b}_j] \Leftrightarrow i <_I j$.

Proof. Add constants $C = \{\bar{c}_i, \bar{d}_i \ : \ i \in I\}$ to the language. Consider the $L(C)$-theory $T' = T \cup \{\varphi(\bar{c}_i, \bar{d}_j) \wedge \neg\varphi(\bar{c}_j, \bar{d}_i) \ : \ i <_I j\}$. Then, T' is satisfiable by the Compactness Theorem 1.11 since φ has the order property. Thus, there is a model $\mathfrak{M}'$ of T', and its L-reduct $\mathfrak{M}$ is a model of T such that $\mathfrak{M} \models \varphi\left[\bar{c}_i^{\mathfrak{M}'}, \bar{d}_j^{\mathfrak{M}'}\right] \Leftrightarrow i < j$. $\square$

Lemma 2.4. *Let T be an L-theory, $\varphi(\overline{x},\overline{y}) \in \mathrm{For}_{\bar{s},\bar{s}'}L$ with the binary tree property and $\mu \geq \omega$ be an ordinal. Then, there are a model $\mathfrak{M} \models T$ and a sequence $(\overline{b}_n)_{n\in{}^{<\mu}2}$ such that, for all $\sigma \in {}^{\omega}2$, the set*

$$\{\varphi^{\sigma(\alpha)}(\overline{x},\overline{b}_{\sigma_{|\alpha}}) \ : \ \alpha \in \mu\}$$

is a type — where $\varphi^0 := \varphi$ and $\varphi^1 := \neg\varphi$.

Proof. Add constants $C = \{\overline{c}_\sigma \ : \ \sigma \in {}^{\mu}2\}$ and $B = \{\overline{b}_n \ : \ n \in {}^{<\mu}2\}$ to the language. Consider the $L(C,B)$-theory $T' = T \cup \{\varphi(\overline{c}_\sigma, b_{\sigma_{|\alpha}}) \ : \ \alpha \in \mu \text{ and } \sigma \in {}^{\mu}2\}$. By the Compactness Theorem 1.11, since φ has the binary tree property, T' is satisfiable. Indeed, any finite collection $B_0 \subseteq B$ could be completed to be embedded in a binary tree: complete every chain appropiately alternating the 0's and 1's to paste the disjoint chains forming a unique tree. Thus, there is a model $\mathfrak{M}'$ of T', and its L-reduct $\mathfrak{M}$ is a model of T such that $\{\varphi^{\sigma(\alpha)}(\overline{x},\overline{b}^{\mathfrak{M}'}_{\sigma_{|\alpha}}) \ : \ \alpha \in \mu\}$ is a type realized by $c^{\mathfrak{M}'}_\sigma$ for each $\sigma \in {}^{\mu}2$. $\square$

Lemma 2.5. ***(Erdös-Makkai)*** *Let B be an infinite set and $\mathcal{S}$ be a set of subsets of B with $\mathrm{card}(B) < \mathrm{card}(\mathcal{S})$. Then, there are sequences $(b_i)_{i\in\omega}$ from B and $(S_j)_{j\in\omega}$ from $\mathcal{S}$ such that $\forall i,j \ \ b_i \in S_j \Leftrightarrow i < j$ or $\forall i,j \ \ b_i \in S_j \Leftrightarrow j < i$.*

Proof. Let $\mathcal{S}' \subset \mathcal{S}$ be such that $\mathrm{card}(\mathcal{S}') = \mathrm{card}(B)$ and every pair of disjoint finite subsets of B separated by an element of $\mathcal{S}$ are also separated by an element of $\mathcal{S}'$. Since $\mathrm{card}(\mathcal{S}') < \mathrm{card}(\mathcal{S})$, there is an element $S^* \in \mathcal{S}$ which is not a boolean combination of elements of $\mathcal{S}'$. We define sequences $(b'_i)_{i\in\omega}$ from S^*, $(b''_i)_{i\in\omega}$ from $B \setminus S^*$ and $(S_j)_{j\in\omega}$ from $\mathcal{S}'$ such that, for every j, S_j separates $\{b'_0,\dots,b'_j\}$ and $\{b''_0,\dots,b''_j\}$, and $b'_i \in S_j \Leftrightarrow b''_i \in S_j$ for any $j < i$. Indeed, assume that there are constructed the sequences $(b'_i)_{i<n}$, $(b''_i)_{i<n}$ and $(S_j)_{j<n}$, I claim that there exist b'_n and b''_n such that $b'_n \in S_j \Leftrightarrow b''_n \in S_j$ for any $j < n$. In other case, we have that

$$S^* = \bigcup_{b\in S^*}\left(\bigcap_{j<n\,:\,b\in S_j} S_j \setminus \bigcup_{j<n\,:\,b\notin S_j} S_i\right).$$

which means that S^* is a boolean combination — note that the union is finite. Chosen b'_n and b''_n, since S^* separates $\{b'_i\}_{i\leq n}$ and $\{b''_i\}_{i\leq n}$, there is an $S_n \in \mathcal{S}'$ separating these sets. The latter concludes by recursion the construction of these sequences. Note that $b''_i \in S_j \Rightarrow j < i$ and $b'_i \in S_j \Rightarrow i < j$. It suffice to prove that there is a subset $I \subseteq \mathbb{N}$ such that $b'_i \in S_j$ for every $j < i$ of I or that $b'_i \notin S_j$ for every $j < i$ of I, because then, in the fist case, $(b''_i)_{i\in I}$ and $(S_j)_{j\in I}$ satisfy $b''_i \in S_j \Leftrightarrow j < i$ and, in the second case, $(b'_i)_{i\in I}$ and $(S_j)_{j\in I}$ satisfy $b'_i \in S_j \Leftrightarrow i < j$. Let us prove the existence of I. Consider $\{(i,j) \in \mathbb{N}^2 \ : \ i < j\}$. We define a strict creasing sequence $(i_k)_{k\in\omega}$ with $i_0 = 0$ and a strict decreasing sequence $(I_k)_{k\in\omega}$ with $I_0 = \mathbb{N}$ of infinite subsets such that, for every $k \in \omega$, $i_k \in I_k$, $I_{k+1} \subseteq \{j \in \mathbb{N} \ : \ i_k < j\}$ and either $b'_{i_k} \in S_j$ for every $j \in I_{k+1}$ or $b'_{i_k} \notin S_j$ for every $j \in I_{k+1}$. Indeed, given $(i_k)_{k\leq K}$ and $(I_k)_{k\leq K}$, by the pigeonhole principle, we have that $\{j \in I_K : j > i_k \text{ and } b_{i_K} \in S_j\}$ is infinite or $\{j \in I_K : j > i_k \text{ and } b_{i_K} \notin S_j\}$ is infinite. Choose the infinite one and let I_{K+1} be this and $i_{K+1} \in I_{K+1}$ be any element. Now, by the pigeonhole principle, $\{k \in \omega : \forall j \in I_{k+1}\ b'_{i_k} \in S_j\}$ is infinite or $\{k \in \omega : \forall i \in I_{k+1}\ b'_{i_k} \notin S_j\}$ is infinite. Choose the infinite one and let K be this. Let $I = \{i_k : k \in K\}$. $\square$

Theorem 2.6. *Let T be an L-theory that has infinite models and $\varphi(\overline{x}, \overline{y}) \in \mathrm{For}_{\overline{s},\overline{s}'}L$. The following are equivalent:*
(1) There is an infinite cardinal λ such that $\mathrm{card}(\mathcal{S}_\varphi^{\mathfrak{M}}(A)) \leq \lambda$ for every $\mathfrak{M} \models T$ and every A sorted subset with $\mathrm{card}(A) \leq \lambda$.
(2) $\mathrm{card}\left(\mathcal{S}_\varphi^{\mathfrak{M}}(A)\right) \leq \sup\{\mathrm{card}(A_s), \mathrm{card}(S) : s \in S\}$ for every $\mathfrak{M} \models T$ and every sorted subset A with infinitely many elements of each sort.
(3) φ does not have the order property.
(4) φ does not have the binary tree property.

Proof. (1)$\Rightarrow$(4) Suppose that φ has the binary tree property. Given $\lambda \geq \mathrm{card}(L)$ infinite cardinal, let μ be minimal such that $2^\mu > \lambda$. By the lemma 2.4, we know that there is a model $\mathfrak{M}$ and a sequence $(b_i)_{i \in {}^{<\mu}2}$ such that, for each $\sigma \in {}^{\mu}2$,

$$\widetilde{q}_\sigma = \{\varphi^{\sigma(\alpha)}(\overline{x}, \overline{b}_{\sigma_{|\alpha}}) : \alpha \in \mu\}$$

is a type. Then, $\mathcal{S}_\varphi^{\mathfrak{M}}(B)$ has cardinal $2^\mu > \lambda$. But $B = \{b_i\}_{i \in {}^{<\mu}2}$ has cardinal less or equal than λ by minimality of μ. The latter negates (1).
(4)$\Rightarrow$(3) Let $I = {}^{\leq\omega}2$ and choose a linear order $<_I$ in I such that, for every $\sigma \in {}^{\omega}2$ and $n \in \omega$, $\sigma <_I \sigma_{|n} \Leftrightarrow \sigma(n) = 1$. By lemma 2.3, if φ has the order property there are a model $\mathfrak{M}$ and sequences $(a_i)_{i \in I}$ and $(b_i)_{i \in I}$ such that $\varphi[a_i, b_j] \Leftrightarrow i <_I j$. Thus, $\mathfrak{M}$ and $(b_i)_{i \in {}^{<\omega}2}$ satisfies the tree property.
(3)$\Rightarrow$(2) Let $\mathfrak{M} \models T$ and A be a sorted subset of infinitely many elements of each sort with $\mathrm{card}(\mathcal{S}_\varphi^{\mathfrak{M}}(A)) > \sup\{\mathrm{card}(A_s), \mathrm{card}(S) : s \in S\}$. Applying the Erdös-Makkai's theorem [lemma 2.5] we obtain sequences $(\overline{a}_i)_{i \in \omega}$ from A and $(S_i)_{i \in \omega}$ from $\mathcal{S}_\varphi^{\mathfrak{M}}(A)$ such that either $\forall i, j\ \ \overline{a}_i \in S_j \Leftrightarrow i < j$ or $\forall i, j\ \ \overline{a}_i \in S_j \Leftrightarrow j < i$. Consider $\mathfrak{M}'$ an elementary extension enough saturated [theorem 1.23] and $(\overline{b}_j)_{j \in \omega}$ such that $\mathfrak{M}' \models \varphi[\overline{b}_j, \overline{a}]$ for every $\overline{a} \in S_j$. Thus, $\mathfrak{M}' \models T$ and either $\mathfrak{M}' \models \varphi[\overline{b}_j, \overline{a}_i] \Leftrightarrow i < j$ or $\mathfrak{M}' \models \varphi[\overline{b}_j, \overline{a}_i] \Leftrightarrow j < i$. In the second case, φ has the order property. In the first case, by lemma 2.3, φ has the order property.
(2)$\Rightarrow$(1) It is clear. $\square$

Theorem 2.7. ***(Characterization of stable theories)*** *Let T be a complete L-theory whose models are infinite. Then, T is stable if and only if there is no formula $\varphi \in \mathrm{For}_{\overline{s},\overline{s}'}L$ with the order property. Moreover, T is stable if and only if T is λ-stable for every λ such that $\lambda^{\mathrm{card}(L)} = \lambda$.*

Proof. ($\Rightarrow$) If T is λ-stable, it is clear that $\mathrm{card}(\mathcal{S}_\varphi^{\mathfrak{M}}(A)) \leq \lambda$ for any φ, any $\mathfrak{M} \models T$ and any sorted subset A with $\mathrm{card}(A) \leq \lambda$. Thus, by the theorem 2.6, φ does not have the order property.
($\Leftarrow$) A type $p \in \mathbf{S}_{\overline{x}}^{\mathfrak{M}}(A)$ is determined by the family $\{S_\varphi(p) \in \mathcal{S}_\varphi^{\mathfrak{M}}(A) : \varphi \in \mathrm{For}_{\overline{x},\overline{y}}L\}$, since $\varphi(\overline{x}, \overline{a}) \in p \Leftrightarrow \overline{a} \in S_\varphi(p)$. Thus, if there is not a formula with

the order property,

$$\mathrm{card}(\mathbf{S}^{\mathfrak{M}}_{\overline{x}}(A)) = \prod_{\varphi\in \mathrm{For}\, L} \mathrm{card}(\mathcal{S}^{\mathfrak{M}}_{\varphi}(A)) \leq \sup\{\mathrm{card}(A_s), \mathrm{card}(S) \ : \ s \in S\}^{\mathrm{card}(L)}.$$

Hence, T is λ-stable for any λ such that $\lambda^{\mathrm{card}(L)} = \lambda$. □

Theorem 2.8. *Let $\lambda \geq \mathrm{card}(L)$ be an infinite cardinal and T a complete L-theory whose models are infinite. Then, T is λ-stable if and only if T^{eq} is λ-stable. In particular, T is stable if and only if T^{eq} is stable.*

Proof. The "if" part is clear. We prove the "only if" part. Let $\mathfrak{M}^{\mathrm{eq}} \models T^{\mathrm{eq}}$ and A be a sorted subset of $\mathfrak{M}^{\mathrm{eq}}$ with $\mathrm{card}(A) \leq \lambda$. Let A^* be a sorted subset of $\mathfrak{M}$ such that $\mathrm{card}(A^*) \leq \lambda$ and every element of A is the equivalence class of a tuple from A^*. By theorem 1.32, we have that for any type $p \in \mathbf{S}^{\mathfrak{M}^{\mathrm{eq}}}_{x}(A)$ there is a unique $\widetilde{p} \in \mathbf{S}^{\mathfrak{M}}_{\overline{x}}(A^*)$ such that an element $[\overline{a}]$ realizes p in an elementary extension if and only if $\overline{a}$ realizes p. Since T is λ-stable, $\mathrm{card}(\mathbf{S}^{\mathfrak{M}}_{\overline{x}}(A^*)) \leq \lambda$, so $\mathrm{card}(\mathbf{S}^{\mathfrak{M}}_{x}(A)) \leq \lambda$. Therefore, T^{eq} is λ-stable. □

2.2 Morley's rank

Let $\mathfrak{M}$ be an $\aleph_0$-saturated L-structure and D a definable subset in $\mathfrak{M}$. We define by induction $\mathrm{MR}(D)$, the *Morley's rank* of D, as follows:

$\mathrm{MR}(D) = -1$	if $D = \emptyset$;
$\mathrm{MR}(D) \geq 0$	if $D \neq \emptyset$;
$\mathrm{MR}(D) \geq \alpha + 1$	if there is $\{D_i\}_{i\in\omega}$ a family of pairwise disjoint definable subsets of D such that $\mathrm{MR}(D_i) \geq \alpha$ for each i, and
$\mathrm{MR}(D) \geq \lambda$	if λ limit and if $\mathrm{MR}(D) \geq \xi$ for every $\xi \in \lambda$.

It can be that $\mathrm{MR}(D) \geq \alpha$ for every $\alpha \in \mathbb{O}\mathrm{n}$, then we say that D *has not Morley's rank* and we write $\mathrm{MR}(D) = \infty$. Given φ an $L(M)$-formula, we define $\mathrm{MR}(\varphi) := \mathrm{MR}(\varphi[\mathfrak{M}])$.

Example. Let D be definable:
1.- $\mathrm{MR}(D) = 0$ if and only if D is finite and non-empty,
2.- $\mathrm{MR}(D) = 1$ if and only if D is infinite but there is not an infinite collection of pairwise disjoint infinite definable subsets.

Morley's rank is a general concept of dimension. It is a generalization of Krull's dimension of algebraic closed fields. This generalization is natural since, in any context, *dimension of a "space" is the "longest chain" of subspaces that you can give inside the space*. In this general situation, the "spaces" are the definable sets and the "chains" are given as we have said.

The condition of $\aleph_0$-saturation is given to have invariance of MR under elementary maps and elementary extensions, as the following results show:

Lemma 2.9. *Let $\mathfrak{M}$ be an $\aleph_0$-saturated L-structure and $\varphi(\overline{x}, \overline{y})$ an L-formula, let $\overline{m}$ and $\overline{m}'$ be tuples with the same type in $\mathfrak{M}$. Then,* $\mathrm{MR}(\varphi(\overline{x}, \overline{m})) = \mathrm{MR}(\varphi(\overline{x}, \overline{m}'))$.

Proof. By symmetry, it suffices to prove by induction on α that $\mathrm{MR}(\varphi(\overline{x}, \overline{m})) \geq \alpha \Rightarrow \mathrm{MR}(\varphi(\overline{x}, \overline{m}')) \geq \alpha$. It is clear for $\alpha = 0$ and α limit. So, it suffices to prove that $\mathrm{MR}(\varphi(\overline{x}, \overline{m})) \geq \alpha + 1$ implies that $\mathrm{MR}(\varphi(\overline{x}, \overline{m}')) \geq \alpha + 1$. Let $(\varphi_i(\overline{x}, \overline{m}_i))_{i \in \omega}$ be a sequence of pairwise disjoint formulas such that each $\varphi_i(\overline{x}, \overline{m}_i)$ implies $\varphi(\overline{x}, \overline{m})$ and $\mathrm{MR}(\varphi_i(\overline{x}, \overline{m}_i)) \geq \alpha$. Now, $\mathfrak{M}$ being $\aleph_0$-saturated is also $\aleph_0$-homogeneous, so, since $\mathrm{tp}(\overline{m}) = \mathrm{tp}(\overline{m}')$, we can choose $(\overline{m}'_i)_{i \in \omega}$ such that $\mathrm{tp}(\overline{m}, \overline{m}_0, \ldots, \overline{m}_k) = \mathrm{tp}(\overline{m}', \overline{m}'_0, \ldots, \overline{m}'_k)$ as follows: given $(\overline{m}'_i)_{i \in n}$, consider the elementary map $f_{n-1} : \overline{m}_i \mapsto \overline{m}'_i$ for $i \in n$ and $f_{n-1}(\overline{m}) = \overline{m}'$. Let f_n be an elementary map extending f_{n-1} to $\overline{m}_n$. Thus, $\overline{m}'_n = f_n(\overline{m}_n)$.

Hence, $(\varphi_i(\overline{x}, \overline{m}'_i))_{i \in \omega}$ is a sequence of pairwise disjoint formulas such that each one implies $\varphi(\overline{x}, \overline{m}')$, and also, by hypothesis of induction, for every $i \in \omega$, $\mathrm{MR}(\varphi_i(\overline{x}, \overline{m}'_i)) \geq \alpha$. □

Lemma 2.10. *Let $\mathfrak{M}$ and $\mathfrak{N}$ be $\aleph_0$-saturated L-structures with $\mathfrak{M} \preceq \mathfrak{N}$ and let φ be an $L(M)$-formula. Then,* $\mathrm{MR}(\varphi[\mathfrak{N}]) = \mathrm{MR}(\varphi[\mathfrak{M}])$.

Proof. It is clear that $\mathrm{MR}(\varphi[\mathfrak{N}]) \geq \mathrm{MR}(\varphi[\mathfrak{M}])$. We prove that $\mathrm{MR}(\varphi[\mathfrak{N}]) \geq \alpha \Rightarrow \mathrm{MR}(\varphi[\mathfrak{M}]) \geq \alpha$ by induction on α. It is clear for $\alpha = 0$ and α limit. Let $\overline{m}$ be the parameters of φ. Let $(\varphi_i(\overline{x}, \overline{n}_i))_{i \in \omega}$ be a sequence of pairwise disjoint formulas such that each $\varphi_i(\overline{x}, \overline{n}_i)$ implies $\varphi(\overline{x}, \overline{m})$ and has $\mathrm{MR}(\varphi_i(\overline{x}, \overline{n}_i)) \geq \alpha$, where $(\overline{n}_i)_{i \in \omega}$ is from N.

Now, $\mathfrak{M}$ and $\mathfrak{N}$ being $\aleph_0$-saturated are also $\aleph_0$-homogeneous, so we can choose $(\overline{m}_i)_{i \in \omega}$ such that $\mathrm{tp}(\overline{n}_0, \ldots, \overline{n}_k / \overline{m}) = \mathrm{tp}(\overline{m}_0, \ldots, \overline{m}_k / \overline{m})$ as follows: given $(\overline{m}_i)_{i \in n}$, consider the elementary map $f_{n-1} : \overline{n}_i \mapsto \overline{m}_i$ for $i \in n$ and $f_{n-1}(\overline{m}) = \overline{m}$. Let g be an elementary map extending f_{n-1} to $\overline{n}_n$ and let $\overline{m}_n$ be from $\mathfrak{M}$ such that $\mathrm{tp}(\overline{m}_n / \overline{m}, \overline{m}_1, \ldots, \overline{m}_{n-1}) = \mathrm{tp}(g(\overline{n}_n) / \overline{m}, g(\overline{n}_1), \ldots, g(\overline{n}_{n-1}))$.

Then, $\alpha = \mathrm{MR}(\varphi_i(\overline{x}, \overline{n}_i)[\mathfrak{N}]) = \mathrm{MR}(\varphi_i(\overline{x}, \overline{m}_i)[\mathfrak{N}])$ for every $i \in \omega$ by the lemma 2.9. By induction hypothesis, $\mathrm{MR}(\varphi_i(\overline{x}, \overline{m}_i)[\mathfrak{M}]) \geq \alpha$ for every $i \in \omega$. We conclude by noting that $(\varphi_i(\overline{x}, \overline{m}_i))_{i \in \omega}$ is a sequence of pairwise disjoint formulas such that each $\varphi_i(\overline{x}, \overline{m}_i)$ implies $\varphi(\overline{x}, \overline{m})$. □

This results allows us to work with Morley's ranks in monster models. If $\mathfrak{M}$ is any infinite L-structure (not necessarily $\aleph_0$-saturated), we define $\mathrm{MR}(\varphi)$ as $\mathrm{MR}(\varphi[\mathfrak{C}])$ where $\mathfrak{C}$ is its monster extension, which is well defined by the above result. Note that, in this case, $\mathrm{MR}(\varphi) = \mathrm{MR}(\varphi[\mathfrak{N}])$ for $\mathfrak{N}$ an $\aleph_0$-saturated elementary extension.

We say that a complete theory whose models are infinite is *totally transcendental* if every definable class of its monster model has Morley's rank.

Remark. Note that the theorem 1.32 implies $\mathrm{MR}^{\mathfrak{M}}(D) = \mathrm{MR}^{\mathfrak{M}^{\mathrm{eq}}}(D)$ for definable sets D of $\mathfrak{M}$.

Now, we are going to prove some basic properties of the Morley's rank.

Proposition 2.11. ***(Fundamental property of Morley's rank)*** *Let D and E be definable subsets in $\mathfrak{M}$. Then,*

$$\mathrm{MR}(D \cup E) = \max\{\mathrm{MR}(D), \mathrm{MR}(E)\}.$$

Proof. Of course, $\mathrm{MR}(D \cup E) \geq \max\{\mathrm{MR}(D), \mathrm{MR}(E)\}$. We prove the other inequality by induction on $\alpha = \mathrm{MR}(D \cup E)$. For α limit or $\alpha = 0$ it is clear. For $\alpha + 1$, given F definable subset of $D \cup E$ with $\mathrm{MR}(F) \geq \alpha$, we have by induction hypothesis that $\mathrm{MR}(F \cap D) \geq \alpha$ or $\mathrm{MR}(F \cap E) \geq \alpha$. Therefore, by the pigeonhole principle, if $(F_i)_{i\in\omega}$ is a sequence of pairwise disjoint definable subsets of $D \cup E$ such that $\mathrm{MR}(F_i) \geq \alpha$, either $(F_i \cap D)_{i\in\omega}$ or $(F_i \cap E)_{i\in\omega}$ has a infinite subsequence of pairwise disjoint definable subsets with Morley's rank greater or equal than α. $\square$

Proposition 2.12. ***(Morley's degree)*** *Let D be a definable subset in $\mathfrak{M}$ of Morley's rank α. Then, there are a natural number $d \in \mathbb{N}^*$ and a definable partition $E_1, \ldots, E_d$ of D such that $\mathrm{MR}(E_i) = \alpha$ and, for every $F \subset E_i$ definable, either $\mathrm{MR}(F) < \alpha$ or $\mathrm{MR}(E_i \setminus F) < \alpha$.*

Moreover, such d is independent of the partition of D, and it is called the Morley's degree *of D and denoted by $\mathrm{Md}(D)$.*

Proof. By contradiction, we define by recursion two sequences of definable subsets $(E_i)_{i\in\omega}$ and $(F_i)_{i\in\omega}$ such that F_{i+1} is disjoint of $E_0, \ldots, E_{i+1}$, $(E_i)_{i\in\omega}$ are pairwise disjoint, $\mathrm{MR}(E_i) = \mathrm{MR}(F_i) = \alpha$, $E_{i+1} \cup F_{i+1} = F_i$ and F_i does not satisfy the theorem for each i. Since D does not satisfy the theorem, there is a definable subset $E_0 \subseteq D$ such that $\mathrm{MR}(E_0) = \mathrm{MR}(D \setminus E_0) = \mathrm{MR}(D)$. Since D does not satisfy the theorem, E_0 or $F_0 = D \setminus E_0$ do not satisfy the theorem. Assume F_0 does not satisfy it. Given $(E_i)_{i\leq n}$ and $(F_i)_{i\leq n}$, since F_n does not satisfy the theorem, there are disjoint definable subsets E_{n+1} and F_{n+1} such that $E_{n+1} \cup F_{n+1} = F_n$, $\mathrm{MR}(E_{n+1}) = \mathrm{MR}(F_{n+1}) = \mathrm{MR}(F_n) = \alpha$ and F_{n+1} does not satisfy the theorem. Of course, E_{n+1} is disjoint of $E_0, \ldots, E_n$ and F_{n+1} is disjoint of $E_0, \ldots, E_n$.

Therefore, $(E_i)_{i\in\omega}$ is a sequence of disjoint definable subsets of D with $\mathrm{MR}(E_i) = \alpha$. The latter implies that $\mathrm{MR}(D) \geq \alpha + 1$, a contradiction.

To prove that d is constant, suppose two partitions $D = X_1 \cup \cdots \cup X_d = Y_1 \cup \cdots \cup Y_{d'}$ with $d < d'$. Then, for each $j \leq d'$, since Y_j satisfies the property of the statement, by the proposition 2.11, there is just one $i \leq d$ such that $\mathrm{MR}(Y_j \cap X_i) = \alpha$. That gives us a contradiction by the pigeonhole principle whenever $d' > d$, since the Y_j's are pairwise disjoint and the X_i's satisfy the property given in the statement. $\square$

We define $\mathrm{Md}_\alpha(D)$ as $\mathrm{Md}(D)$ when $\mathrm{MR}(D) = \alpha$, as 0 when $\mathrm{MR}(D) < \alpha$ and as ∞ when $\mathrm{MR}(D) > \alpha$. Also, given an $L(M)$-formula φ, we define $\mathrm{Md}_\alpha(\varphi) := \mathrm{Md}_\alpha(\varphi(\mathfrak{M}))$ and $\mathrm{Md}(\varphi) := \mathrm{Md}(\varphi(\mathfrak{M}))$.

Corollary 2.13. *Let α be a cardinal and D and E two disjoint definable subsets in $\mathfrak{M}$. Then, $\mathrm{Md}_\alpha(D \cup E) = \mathrm{Md}_\alpha(D) + \mathrm{Md}_\alpha(E)$.*

Lemma 2.14. ***(Maximal Morley's rank)*** *Let* $\mathfrak{C}$ *be a monster model over* L. *Then, there exists an ordinal* $\alpha \in \mathbb{O}\mathrm{n}$ *such that, for every definable class* $\mathbf{D}$ *of* $\mathfrak{C}$, $\mathrm{MR}(\mathbf{D}) = \infty$ *if and only if* $\mathrm{MR}(\mathbf{D}) > \alpha$.

Proof. It suffices to prove that $X = \{\mathrm{MR}(\mathbf{D}) : \mathbf{D} \text{ is definable}\}$ is a set, because then, $\alpha = \sup(X \cap \mathbb{O}\mathrm{n})$. We have that $X = \{\mathrm{MR}(\varphi(\overline{x}, \overline{c})) : \varphi \in \mathrm{For}\, L \text{ and } \overline{c} \text{ from } \mathfrak{C}\}$. Now, if $\mathrm{tp}(\overline{c}) = \mathrm{tp}(\overline{d})$, then $\mathrm{MR}(\varphi(\overline{x}, \overline{c})) = \mathrm{MR}(\varphi(\overline{x}, \overline{d}))$ by the lemma 2.9. Choose a set $\{\overline{c}_p : p \in \bigcup_{\overline{s} \in {}^{<\omega}S} \mathbf{S}^{\mathfrak{C}}_{\overline{s}}(\emptyset)\}$ such that $\mathfrak{C} \models p[\overline{c}_p]$ for each p type. Then,

$$X = \left\{ \mathrm{MR}(\varphi(\overline{x}, \overline{c}_p)) \,:\, p \in \bigcup_{\overline{s} \in {}^{<\omega}S} \mathbf{S}^{\mathfrak{C}}_{\overline{s}}(\emptyset) \text{ and } \varphi \in \mathrm{For}\, L \right\},$$

so it is a set. □

Corollary 2.15. *Let* D *be a definable set of* $\mathfrak{M}$ *with* $\mathrm{MR}(D) = \infty$. *Then, there is a definable subset* $D' \subseteq D$ *such that* $\mathrm{MR}(D') = \infty$ *and* $\mathrm{MR}(D \setminus D') = \infty$. *Moreover, the same is true for definable classes in monster models.*

Corollary 2.16. ***(Binary tree property in definable sets without Morley's rank)*** *Let* D *be a definable set of* $\mathfrak{M}$ *with* $\mathrm{MR}(D) = \infty$. *Then, there is a binary tree* $(D_w)_{w \in {}^{<\omega}2}$ *of definable subsets of* D *such that* $\mathrm{MR}(D_w) = \infty$ *for each* $w \in {}^{<\omega}2$. *Moreover, the same is true for definable classes in monster models.*

Corollary 2.17. *Let* T *be an* L*-theory. Then,* T *is totally transcendental if and only if* T *does not have a binary tree.*

Proof. The "if" part is clear. On the other hand, if T has a binary tree $(D_w)_{w \in {}^{<\omega}2}$, let $\alpha = \min\{\mathrm{MR}(D_w) : w \in {}^{<\omega}2\}$ and let $d = \min\{\mathrm{Md}_\alpha(D_w) : w \in {}^{<\omega}2\}$. Let D_w be such that $\mathrm{Md}_\alpha(D_w) = d$. Then, $2d \geq \mathrm{Md}_\alpha(D_{w \cup \{(n,0)\}}) + \mathrm{Md}_\alpha(D_{w \cup \{(n,1)\}}) = d$. □

Theorem 2.18. *Let* T *be an* ω*-stable* L*-theory. Then,* T *is totally transcendental.*

Proof. Let $\mathfrak{C}$ be the monster model of T. If T is not totally transcendental, by the corollary 2.16, we have that there is a tree $(\mathbf{D}_w)_{w \in {}^{<\omega}2}$ such that $\mathrm{MR}(\mathbf{D}_w) = \infty$, for each $w \in {}^{<\omega}2$. Let A_w be finite sorted set such that $\mathbf{D}_w$ is A_w-definable. Let $A = \bigcup_{w \in {}^{<\omega}2} A_w$. Then, $\mathrm{card}(A) \leq \aleph_0$. For each $\sigma \in {}^{\omega}2$, let p_σ be a complete type over A such that $\{\underline{\mathbf{D}_{\sigma_{|n}}} : n \in \omega\} \subseteq p_\sigma$. Then, $\{p_\sigma\}_{\sigma \in {}^{\omega}2} \subseteq \mathbf{S}^{\mathfrak{C}}(A)$. Since $p_\sigma \neq p_{\sigma'}$ if $\sigma \neq \sigma'$, we conclude that $\mathbf{S}^{\mathfrak{C}}(A)$ is uncountable. Therefore, T is not ω-stable. □

Theorem 2.19. ***(The order property in definable sets with Morley's rank)*** *Let* $\mathbf{D}$ *be a definable class with Morley's rank in a monster model* $\mathfrak{C}$. *Then, there is no formula* $\varphi \in \mathrm{For}_{\overline{s}, \overline{s}'}\, L$ *and sequences* $(\overline{a}_i)_{i \in \omega}$ *and* $(\overline{b}_i)_{i \in \omega}$ *with* $\overline{a}_i \in \mathbf{D}$, *for each* $i \in \omega$, *such that*

$$\mathfrak{C} \models \varphi[\overline{a}_i, \overline{b}_j] \Leftrightarrow i < j.$$

In particular, totally transcendental theories are stable.

Proof. By lemma 2.3, if there is such formula φ and such sequences, there are also $(\overline{a}_i)_{i\in\mathbb{Q}}$ and $(\overline{b}_i)_{i\in\mathbb{Q}}$ such that $\mathfrak{C}\models\varphi[\overline{a}_i,\overline{b}_j]\Leftrightarrow i<j$. Let $A=\{\overline{a}_i\}_{i\in\mathbb{Q}}$ and $B=\{\overline{b}_i\}_{i\in\mathbb{Q}}$. If $\mathrm{MR}(\mathbf{D})\in\mathbb{O}\mathrm{n}$, there exists

$$\alpha=\min\{\mathrm{MR}(\phi)\ :\ \phi\in\mathrm{For}_{\overline{x}}L(A,B),\ \phi[\mathfrak{C}]\subseteq\mathbf{D}\text{ and }I_\phi\text{ open interval}\},\text{ and}$$
$$d=\min\{\mathrm{Md}_\alpha(\phi)\ :\ \phi\in\mathrm{For}_{\overline{x}}L(A,B),\ \phi[\mathfrak{C}]\subseteq\mathbf{D}\text{ and }I_\phi\text{ open interval}\},$$

where $I_\phi=\{i\in\mathbb{Q}\,:\,\mathfrak{C}\models\phi[\overline{a}_i]\}$. Let $\phi\in\mathrm{For}_{\overline{x}}L(A,B)$ be such that $\mathrm{MR}(\phi)=\alpha$ and $\mathrm{Md}(\phi)=d$. Let $j\in I_\phi$, then $\phi'=\phi(\overline{x})\wedge\varphi(\overline{x},b_j)$ and $\phi''=\phi(\overline{x})\wedge\neg\varphi(x,b_j)$ are such that $\mathrm{MR}(\phi')=\mathrm{MR}(\phi'')=\alpha$ by minimality of α and $d=\mathrm{Md}(\phi)=\mathrm{Md}(\phi')+\mathrm{Md}(\phi'')\geq 2d$, a contradiction. $\square$

2.3 Morley's rank of types

Let $A\subseteq M$ be a sorted set of parameters and $\Sigma(\overline{x})$ a type. The *Morley's rank of the type* $\Sigma(\overline{x})$ is $\mathrm{MR}(\Sigma):=\min\{\mathrm{MR}(\varphi)\ :\ \varphi\in\Sigma(\overline{x})\}$ and the *Morley's degree of the type* $\Sigma(\overline{x})$ is $\mathrm{Md}(\Sigma):=\mathrm{Md}_{\mathrm{MR}(\Sigma)}(\Sigma)$ where $\mathrm{Md}_\alpha(\Sigma):=\min\{\mathrm{Md}_\alpha(\varphi)\ :\ \varphi\in\Sigma\}$ for any $\alpha\in\mathbb{O}\mathrm{n}$. We will work only with complete types in this section.

We also define the *Morley's rank of a tuple* $\overline{a}$ over a sorted set $A\subseteq M$ as $\mathrm{MR}(a/A):=\mathrm{MR}(\mathrm{tp}(a/A))$, and in the same way we define $\mathrm{Md}(a/A)$ and $\mathrm{Md}_\alpha(a/A)$. We have analogous definitions for monster models, even for types over classes.

Proposition 2.20. ***(Description of a type with Morley's rank)*** *Let $p\in\mathbf{S}^{\mathfrak{M}}(A)$ and $\phi\in p$ such that $\mathrm{MR}(\phi)=\mathrm{MR}(p)$ and $\mathrm{Md}(\phi)=\mathrm{Md}(p)$. Then,*

$$p=\{\psi\in\mathrm{For}\ L(A)\ :\ \mathrm{MR}(\phi\wedge\neg\psi)<\mathrm{MR}(p)\}\,.$$

Proof. Let $\alpha=\mathrm{MR}(p)$, let $\Sigma=\{\psi\in\mathrm{For}L(A)\,:\,\mathrm{MR}(\phi\wedge\neg\psi)<\alpha\}$ and let ψ be an $L(A)$-formula. First, if $\psi\notin p$, then $\neg\psi\in p$ and $\phi\wedge\neg\psi\in p$, so $\mathrm{MR}(\phi\wedge\neg\psi)\geq\alpha$ and $\psi\notin\Sigma$. On the other hand, when $\psi\notin\Sigma$, if $\psi\in p$, then $\mathrm{MR}(\phi\wedge\neg\psi)\geq\alpha$ and $\mathrm{MR}(\phi\wedge\psi)\geq\alpha$. Therefore, $\mathrm{MR}(\phi)=\max\{\mathrm{MR}(\phi\wedge\neg\psi),\mathrm{MR}(\phi\wedge\psi)\}=\alpha$, so $\mathrm{MR}(\phi\wedge\neg\psi)=\mathrm{MR}(\phi\wedge\psi)=\alpha$ and $2\mathrm{Md}(\phi)\leq\mathrm{Md}_\alpha(\phi\wedge\neg\psi)+\mathrm{Md}_\alpha(\phi\wedge\psi)=\mathrm{Md}(\phi)$, a contradiction. $\square$

Proposition 2.21. *Let $\phi\in\mathrm{For}\,L(A)$ be consistent and such that there is not a non-trivial finite A-definable partition of ϕ in formulas with the same Morley's rank. Then, $p=\{\psi\in\mathrm{For}\ L(A)\,:\,\mathrm{MR}(\phi\wedge\neg\psi)<\alpha\}$ is a complete type satisfying $\mathrm{MR}(p)=\mathrm{MR}(\phi)$ and $\mathrm{Md}(p)=\mathrm{Md}(\phi)$.*

Proof. Of course, $\phi\in p$, and $\psi\in p$ implies $\psi[\mathfrak{M}]\neq\emptyset$. On the other hand, since $\mathrm{MR}(\phi\wedge(\neg\psi_1\vee\neg\psi_2))=\max\{\mathrm{MR}(\phi\wedge\neg\psi_1),\mathrm{MR}(\phi\wedge\neg\psi_2)\}$, we have that p is closed under $\wedge$. So, p is a partial type. To prove that p is a complete type, it suffices to see that, for any $\psi\in\mathrm{For}\ L(A)$, either $\psi\in p$ or $\neg\psi\in p$. However, the latter is clear since $\phi\wedge\neg\psi$ and $\phi\wedge\psi$ together form an A-definable partition of ϕ, so one has Morley's rank less than α or both ones have Morley's degree less than $\mathrm{Md}(\phi)$. Finally, given $\psi\in p$, $\mathrm{MR}(\psi\wedge\phi)=\mathrm{MR}(\phi)$ and $\mathrm{MR}(\neg\psi\wedge\phi)<\alpha$. So, $\mathrm{Md}(\phi)=\mathrm{Md}(\psi\wedge\phi)+\mathrm{Md}_{\mathrm{MR}(\phi)}(\phi\wedge\neg\psi)=\mathrm{Md}(\psi\wedge\phi)$. Thus, $\mathrm{MR}(\phi)=\mathrm{MR}(\psi\wedge\phi)\leq\mathrm{MR}(\psi)$ and, if $\mathrm{MR}(\psi)=\alpha$, $\mathrm{Md}(\phi)=\mathrm{Md}(\psi\wedge\phi)\leq\mathrm{Md}(\psi)$. $\square$

Let D be an A-definable set in $\mathfrak{M}$ with Morley's rank. A *generic type* over A in D is a type p over A in $\mathfrak{M}$ such that $D \in p$ and $\mathrm{MR}(D) = \mathrm{MR}(p)$. A *generic element* of D over A is an element $a \in D$ which realizes a generic type of D over A.

Note that if D has Morley's rank and is A-definable, there are always generic types over A by the propositions 2.12 and 2.21, so

$$\mathrm{MR}(D) = \max\{\mathrm{MR}(p) \ : \ p \in \mathbf{S}^{\mathfrak{M}}(A) \text{ generic in } D\}.$$

Moreover, it is clear that

$$\mathrm{Md}(D) = \sum \left\{\mathrm{Md}(p) \ : \ p \in \mathbf{S}^{\mathfrak{M}}(A) \text{ generic of } D\right\}.$$

So, in particular, the number of generic types is always less or equal than $\mathrm{Md}(D)$. To have generic elements we need enough saturation.

Theorem 2.22. *Let T be a totally transcendental L-theory. Then, T is λ-stable for every $\lambda \geq \mathrm{card}(L)$.*

Proof. Let $\mathfrak{C}$ be the monster model. If T is totally transcendental, we have that every $p \in \mathbf{S}^{\mathfrak{C}}_x(A)$ is given by a formula $\phi \in \mathrm{For}_x L(A)$ as $p = \{\psi \in \mathrm{For}_x L(A) \ : \ \mathrm{MR}(\phi \wedge \neg\psi) < \mathrm{MR}(p)\}$ [proposition 2.20]. Thus, by the axiom of choice, we have an one-to-one function from $\mathbf{S}^{\mathfrak{C}}_x(A)$ to $\mathrm{For}_x L(A)$. If $\mathrm{card}(A) \leq \lambda$ and $\mathrm{card}(L) \leq \lambda$, then there must be $\mathrm{card}\left(\mathbf{S}^{\mathfrak{C}}_x(A)\right) \leq \lambda$. Therefore, T is λ-stable. $\square$

Theorem 2.23. ***(Morley's rank of algebraic elements)*** *Let A be a sorted subset and $\overline{b}$ and $\overline{a}$ from $\mathfrak{M}$. Then, if $\overline{b}$ is algebraic over $A, \overline{a}$, $\mathrm{MR}(\overline{b}/A) \leq \mathrm{MR}(\overline{a}/A)$.*

Proof. Let $\mathfrak{C}$ be the monster extension of $\mathfrak{M}$. To simplify the notation, add A to the language and assume that there are not parameters. We prove that $\mathrm{MR}(\overline{b}) \geq \alpha \Rightarrow \mathrm{MR}(\overline{a}) \geq \alpha$ by induction on α. For $\alpha = 0$ or α limit, it is clear. Let $\alpha = \beta + 1$. Assume that $\mathrm{MR}(\overline{b}) \geq \beta + 1$ and $\mathrm{MR}(\overline{a}) \leq \beta$ to aim for a contradiction. Since $\mathrm{MR}(\overline{b}) > \beta$, by induction hypothesis, $\mathrm{MR}(\overline{a}) = \beta$. Let $\mathbf{D}$ be a definable class such that $\overline{a} \in \mathbf{D}$ and $\mathrm{MR}(\overline{a}) = \mathrm{MR}(\mathbf{D}) = \beta$. Let $\phi \in \mathrm{For}_{\overline{x}\overline{y}} L(A)$ be such that $\overline{b} \in \phi(\overline{a}, \overline{y})[\mathfrak{C}]$, $\phi(\overline{a}, \overline{y})[\mathfrak{C}]$ is finite of cardinal k and $\phi(\overline{a}', \overline{y})[\mathfrak{C}]$ has cardinal k for every $\overline{a}'$. Consider $\underline{\mathbf{E}}(\overline{y}) = \exists \overline{x}(\underline{\mathbf{D}}(\overline{x}) \wedge \phi(\overline{x}, \overline{y}))$. Then, $\overline{b} \in \mathbf{E}$, so $\underline{\mathbf{E}} \in \mathrm{tp}(\overline{b})$. Thus, $\mathrm{MR}(\mathbf{E}) \geq \beta + 1$. Let $(\mathbf{E}'_j)_{j \in \omega}$ be a sequence of pairwise disjoint definable subclasses of $\mathbf{E}$ such that $\mathrm{MR}(\mathbf{E}'_j) \geq \beta$. For each $\mathbf{E}'_j$, consider a subclass $\mathbf{E}_j$ with same Morley's rank and with Morley's degree 1 [proposition 2.12]. Let

$$\underline{\mathbf{D}_i}(\overline{x}) = \exists \overline{y}\left(\underline{\mathbf{D}}(\overline{x}) \wedge \underline{\mathbf{E}_i}(\overline{y}) \wedge \phi(\overline{x}, \overline{y})\right)$$

and consider the sequence $(\mathbf{D}_i)_{i \in \omega}$ of definable subclasses of $\mathbf{D}$. I claim that $\mathrm{MR}(\mathbf{D}_i) = \beta$. Indeed, let $(\overline{b}_i)_{i \in \omega} \in \prod_{i \in \omega} \mathbf{E}_i$ be a sequence generic elements in $\mathfrak{C}$. Since $(\overline{b}_i)_{i \in \omega} \in {}^{\omega}\mathbf{E}$, there is a sequence $(\overline{a}_i)_{i \in \omega} \in {}^{\omega}\mathbf{D}$ such that $\mathfrak{C} \models \phi[\overline{a}_i, \overline{b}_i]$ for each i. Then, note that $\overline{a}_i \in \mathbf{D}_i$ and $\overline{b}_i$ is algebraic over $\overline{a}_i$ for each i. By induction hypothesis, $\mathrm{MR}(\overline{a}_i) \geq \beta$, so $\mathrm{MR}(\mathbf{D}_i) \geq \beta$ for each i. Let $\mathbf{D} = \mathbf{D}^1 \cup \cdots \cup \mathbf{D}^d$ be a partition of $\mathbf{D}$ in definable classes of Morley's rank $\mathrm{MR}(\mathbf{D})$ and Morley's degree 1 [proposition 2.12]. For every $i \in \omega$, there is a $k \in \{1, \ldots, d\}$ such that $\mathrm{MR}(\mathbf{D}^k \setminus \mathbf{D}) < \beta$. By the pigeonhole principle, there is a $k \in \{1, \ldots, d\}$ and an infinite set $I_k \subseteq \omega$ such that $(\mathbf{D}_i)_{i \in I_k}$ satisfy that $\mathrm{MR}(\mathbf{D}^k \setminus \mathbf{D}_i) < \beta = \mathrm{MR}(\mathbf{D}^k)$. Consider $q = \{\psi \in \mathrm{For}_x L(A) : \mathrm{MR}(\mathbf{D}^k \setminus \psi[\mathfrak{C}]) < \beta\}$ where A is a sorted set such that $\mathbf{D}^k$ is A-definable and $\overline{\mathbf{D}_i}$ is an A-formula for each $i \in I_k$. By the lemma 2.21, since $\mathrm{MR}(\mathbf{D}^k) = \beta$ and $\mathrm{Md}(\mathbf{D}^k) = 1$, q is a type. By saturation of $\mathfrak{C}$, there is an element realizing q. Since $\overline{\mathbf{D}_i} \in q$ for each $i \in I_k$, we conclude that $\bigcap_{i \in I_k} \mathbf{D}_i \neq \emptyset$. Let $\overline{a}' \in \bigcap_{i \in I_k} \mathbf{D}_i$. Thus, there is a sequence $(\overline{b}_i')_{i \in I_k} \in \prod_{i \in I_k} \mathbf{E}_i$ such that $\mathfrak{C} \models \phi[\overline{a}', \overline{b}_i']$ for each $i \in I_k$. Since $(\mathbf{E}_i)_{i \in I_k}$ are pairwise disjoint, $\{\overline{b}_i'\}_{i \in I_k} \subseteq \phi(\overline{a}', \overline{y})[\mathfrak{C}]$ has cardinal $\aleph_0$, a contradiction since $\phi(\overline{a}', \overline{y})[\mathfrak{C}]$ is finite. □

Corollary 2.24. ***(Morley's rank and definable functions)*** *Let D and E be definable subsets in $\mathfrak{M}$ and $f : D \to E$ a definable onto function in $\mathfrak{M}$. Then, $\mathrm{MR}(E) \leq \mathrm{MR}(D)$.*

Proof. We may assume $\mathrm{Md}(D) = 1$, otherwise take its partition. Let $A \subseteq M$ be a finite sorted set such that D, E and f are A-definable. Then, $\mathrm{MR}(D) = \max\{\mathrm{MR}(d/A) : d \in D\}$ and

$$\mathrm{MR}(E) = \max\{\mathrm{MR}(e/A) : e \in E\} = \max\{\mathrm{MR}(f(d)/A) : d \in D\}.$$

Since f is an A-definable, $f(d) \in \mathrm{dcl}(d, A)$. By the theorem 2.23, $\mathrm{MR}(f(d)/A) \leq \mathrm{MR}(d/A)$. □

Corollary 2.25. *Let D and E be definable subsets in $\mathfrak{M}$ and $f : D \to E$ a definable bijection in $\mathfrak{M}$. Then, $\mathrm{MR}(E) = \mathrm{MR}(D)$ and $\mathrm{Md}(E) = \mathrm{Md}(D)$.*

Corollary 2.26. *Let T be a totally transcendental L-theory. Then, T^{eq} is totally transcendental.*

Proof. Apply the last corollary to the projections

$$\pi_{E_1}^{\mathfrak{C}^{\mathrm{eq}}} \times \cdots \times \pi_{E_n}^{\mathfrak{C}^{\mathrm{eq}}} : \mathbf{C}_{s_1^1} \times \cdots \times \mathbf{C}_{s_{m_n}^n} \to {}^{\mathbf{C}_{s_1^1} \times \cdots \mathbf{C}_{s_{m_1}^1}}/_{E_1} \times \cdots \times {}^{\mathbf{C}_{s_1^n} \times \cdots \mathbf{C}_{s_{m_n}^n}}/_{E_n}.$$

□

Lemma 2.27. *Let $\mathfrak{C}$ be the monster extension of $\mathfrak{M}$ and $\mathbf{D}$ a definable class such that $\mathrm{MR}(\mathbf{D}) = \alpha \in \mathbb{O}\mathrm{n}$. Then, there is an element of $\mathbf{D}$ from $\mathfrak{M}$, provided there is an M-definable class $\mathbf{D}'$ such that $\mathbf{D} \subseteq \mathbf{D}'$ and $\mathrm{MR}(\mathbf{D}) = \mathrm{MR}(\mathbf{D}')$.*

Proof. We may assume that $\mathrm{Md}(\mathbf{D}) = \mathrm{MR}(\mathbf{D}') = 1$. Indeed, by the theorem 2.12, there are two partitions $\mathbf{D} = \mathbf{D}_1 \cup \cdots \cup \mathbf{D}_d$ and $\mathbf{D}' = \mathbf{D}_1' \cup \cdots \cup \mathbf{D}_k'$ such that $\mathrm{Md}_\alpha(\mathbf{D}_i) = \mathrm{Md}_\alpha(\mathbf{D}_j') = 1$ and $\mathbf{D}_j'$ is M-definable, for each i, j. Since $\mathbf{D}_1 = (\mathbf{D}_1 \cap \mathbf{D}_j') \cup \cdots \cup (\mathbf{D}_1 \cap \mathbf{D}_j')$ and $\mathrm{Md}(\mathbf{D}_1) = 1$, by the propositions 2.11 and 2.13, there is just one $j \leq k$ such that $\mathrm{MR}(\mathbf{D}_1 \cap \mathbf{D}_j') = \mathrm{MR}(D_1)$. Then, we may consider $\mathbf{D}'_j$ instead of $\mathbf{D}'$ and $\mathbf{D}_1 \cap \mathbf{D}'_j$ instead of $\mathbf{D}$. So assume that $\mathrm{Md}(\mathbf{D}) = \mathrm{Md}(\mathbf{D}') = 1$. We prove the lemma by induction on α. For $\alpha = 0$, since $\mathrm{Md}(\mathbf{D}) = 1 = \mathrm{Md}(\mathbf{D}') = 1$, we have that $\mathbf{D} = \{\overline{d}\}$ and $\mathbf{D}' = \{\overline{d}\}$, so $\overline{d}$ is from $\mathfrak{M}$ since $\mathbf{D}'$ is M-definable. Now, if $\mathrm{MR}(\mathbf{D}) = \mathrm{MR}(\mathbf{D}') = \alpha > 0$, since $\mathrm{Md}(\mathbf{D}') = 1$, it is clear that $\mathrm{MR}(\mathbf{D}' \setminus \mathbf{D}) < \alpha$. Let $\mathrm{MR}(\mathbf{D}' \setminus \mathbf{D}) = \beta$. If $\beta = -1$, then $\mathbf{D}' = \mathbf{D}$ and the lemma follows. If $\beta \geq 0$, because $\mathrm{MR}(\mathbf{D}') = \alpha > \beta$, there is a sequence $(\mathbf{D}'_j)_{j\in\omega}$ of pairwise disjoint M-definable subclasses of $\mathbf{D}'$ such that $\beta \leq \mathrm{MR}(\mathbf{D}'_j) < \alpha$. But, for some $j \in \omega$, $\mathrm{MR}((\mathbf{D}'_j \setminus \mathbf{D})) < \beta$ because $\mathbf{D}' \setminus \mathbf{D}$ has Morley's rank β. Since $\mathrm{MR}(\mathbf{D}'_j \setminus \mathbf{D}) < \beta \leq \mathrm{MR}(\mathbf{D}'_j)$,

$$\mathrm{MR}(\mathbf{D} \cap \mathbf{D}'_j) = \max\{\mathrm{MR}(\mathbf{D} \cap \mathbf{D}'_j), \mathrm{MR}(\mathbf{D}'_j \setminus \mathbf{D})\} = \mathrm{MR}(\mathbf{D}'_j) < \alpha$$

and $\mathbf{D} \cap \mathbf{D}'_j \subseteq \mathbf{D}'_j$. By induction hypothesis, there is an element of $\mathbf{D} \cap \mathbf{D}'_j \subseteq \mathbf{D}$ from $\mathfrak{M}$. $\square$

Theorem 2.28. *Let D be a definable subset in $\mathfrak{M}$ with Morley's rank $\alpha \in \mathbb{O}\mathrm{n}$ and $\phi(\overline{x}, \overline{y})$ an L-formula. Then,*

$$X = \left\{\overline{b} \,:\, \mathrm{MR}\left(D \setminus \phi(\overline{x}, \overline{b})[\mathfrak{M}]\right) < \alpha\right\}$$

is definable with the parameters of D.

Proof. Let $\mathfrak{C}$ be the monster extension of $\mathfrak{M}$ and $\mathbf{D} = \underline{D}[\mathfrak{C}]$. Let

$$\mathbf{X} = \left\{\overline{b} \text{ in } \mathfrak{C} \,:\, \mathrm{MR}(\mathbf{D} \setminus \phi(\overline{x}, \overline{b})[\mathfrak{C}]) < \alpha\right\}.$$

We may assume that $\mathrm{Md}(\mathbf{D}) = 1$, because if $\mathbf{D} = \mathbf{D}_1 \cup \cdots \cup \mathbf{D}_d$ is a partition of $\mathbf{D}$ with $\mathrm{Md}_\alpha(\mathbf{D}_i) = 1$ for each i, then

$$\left\{\overline{b} \,:\, \mathrm{MR}\left(\mathbf{D} \setminus \phi(\overline{x}, \overline{b})[\mathfrak{C}]\right) < \alpha\right\} = \bigcap_{i=1}^{d} \left\{\overline{b} \,:\, \mathrm{MR}\left(\mathbf{D}_i \setminus \phi(\overline{x}, \overline{b})[\mathfrak{C}]\right) < \alpha\right\}.$$

So assume that $\mathrm{Md}(\mathbf{D}) = 1$. I claim that for any formula $\psi(\overline{x}, \overline{y})$ and for every $\overline{c}$ such that $\mathrm{MR}(\mathbf{D} \setminus \psi(\overline{x}, \overline{c})[\mathfrak{C}]) < \alpha$, there exists a finite set $\Delta \subseteq \mathbf{D} \cap \psi(\overline{x}, \overline{c})[\mathfrak{C}]$ from $\mathfrak{M}$ such that for every $\overline{b}$

$$\Delta \subseteq \psi(\overline{x}, \overline{b})[\mathfrak{C}] \Rightarrow \mathrm{MR}(\mathbf{D} \setminus \psi(\overline{x}, \overline{b})[\mathfrak{C}]) < \alpha.$$

Indeed, for any ψ, suppose that for some $\overline{c}$ there is not such Δ. Thus, we can define by recursion two sequences $(\overline{a}_i)_{i\in\omega}$ from $\mathbf{D}$ and $(\overline{b}_j)_{j\in\omega}$ such that

$$\mathfrak{C} \models \psi[\overline{a}_i, \overline{b}_j] \Leftrightarrow i < j,$$

as follows: by the lemma 2.27, there is an $\overline{a}_{-1} \in \mathbf{D} \cap \psi(\overline{x}, \overline{c})[\mathfrak{C}]$ from $\mathfrak{M}$. Since we are supposing that $\Delta_{-1} = \{\overline{a}_{-1}\}$ does not satisfy the claim, there is an element $\overline{b}_0$ such that $\overline{b}_0 \in \psi(\overline{a}_{-1}, \overline{y})[\mathfrak{C}]$ and $\mathrm{MR}(\mathbf{D} \setminus \psi(\overline{x}, \overline{b}_0)[\mathfrak{C}]) = \alpha$. Now, Let $(\overline{a}_i)_{i\in\{-1,\ldots,n-1\}}$ and $(\overline{b}_j)_{j\leq n}$ be already defined. Since $\mathrm{Md}_\alpha(\mathbf{D}) = 1$,

$$\forall j \leq n \ \mathrm{MR}(\mathbf{D} \setminus \psi(\overline{x}, \overline{b}_j)[\mathfrak{C}]) < \alpha \Leftrightarrow \forall j \leq n \ \mathrm{MR}(\mathbf{D} \cap \psi(\overline{x}, \overline{b}_j)[\mathfrak{C}]) = \alpha \Rightarrow$$

$$\mathrm{MR}(\mathbf{D}\bigcup_{j\leq n}\psi(\overline{x},\overline{b}_j)[\mathfrak{C}])=\alpha\Leftrightarrow\mathrm{MR}\left(\mathbf{D}\setminus\bigcup_{j\leq n}\psi(\overline{x},\overline{b}_j)[\mathfrak{C}]\right)<\alpha.$$

By the lemma 2.27, there is an $\overline{a}_n\in\left(\mathbf{D}\setminus\bigcup_{j\leq n}\psi(\overline{x},\overline{b}_j)[\mathfrak{C}]\right)\cap\psi(\overline{x},\overline{c})[\mathfrak{C}]$ from $\mathfrak{M}$. Since we are supposing that $\Delta_n=\{\overline{a}_{-1},\overline{a}_0,\dots,\overline{a}_n\}$ does not satisfy the claim, there is a $\overline{b}_{n+1}$ such that $\mathrm{MR}(\mathbf{D}\setminus\psi(\overline{x},\overline{b}_{n+1})[\mathfrak{C}])<\alpha$ and $\Delta_n\subseteq\psi(\overline{x},\overline{b}_{n+1})[\mathfrak{C}]$. Therefore, $(\overline{a}_i)_{i\in\omega}$ from $\mathbf{D}$ and $(\overline{b}_j)_{j\in\omega}$ are such that $\mathfrak{C}\models\psi[\overline{a}_i,\overline{b}_j]\Leftrightarrow i<j$, a contradiction because of lemma 2.19.

Let $W_1\subseteq\mathcal{P}(M_{s_1}\times\cdots\times M_{s_n})$ be the set of all the finite $\Delta\subseteq\mathbf{D}$ from $\mathfrak{M}$ which satisfy $\Delta\subseteq\varphi(\overline{x},\overline{b})[\mathfrak{C}]\Rightarrow\mathrm{MR}(\mathbf{D}\setminus\varphi(\overline{x},\overline{b})[\mathfrak{C}])<\alpha$. Let $W_2\subseteq\mathcal{P}(M_{s_1}\times\cdots\times M_{s_n})$ be the similar set for $\neg\varphi$. For each $\Delta\in W_1$, let $\chi_\Delta(\overline{y})\in\mathrm{For}_{\overline{y}}L(M)$ be the formula $\chi_\Delta(\overline{y})=\bigwedge_{\overline{a}\in\Delta}\varphi(\overline{a},\overline{y})$. Let $\chi'_{\Delta'}(\overline{y})$ be the similar formula for $\neg\varphi$ and $\Delta'\in W_2$. Then, $\langle\chi_\Delta\rangle\subseteq\{\mathrm{tp}(\overline{b}/M)\,:\,\overline{b}\in\mathbf{X}\}$ and, for each $\overline{b}\in\mathbf{X}$, there is a $\Delta\in W_1$ such that $\chi_\Delta(\overline{b})$. Therefore,

$$\{\mathrm{tp}(\overline{b}/M)\,:\,\overline{b}\in\mathbf{X}\}=\bigcup_{\Delta\in W_1}\langle\chi_\Delta\rangle.$$

Since $\mathrm{Md}(\mathbf{D})=1$, $\overline{b}\notin\mathbf{X}$ if and only if $\mathrm{MR}(\mathbf{D}\setminus\neg\varphi(\overline{x},\overline{b})[\mathfrak{C}])<\alpha$. Then,

$$\{\mathrm{tp}(\overline{b}/M)\,:\,\overline{b}\notin\mathbf{X}\}=\bigcup_{\Delta'\in W_2}\langle\chi'_{\Delta'}\rangle.$$

Thus,

$$\mathbf{S}^{\mathfrak{C}}_{\overline{y}}(M)=\bigcup_{\Delta\in W_1}\langle\chi_\Delta\rangle\cup\bigcup_{\Delta'\in W_2}\langle\chi'_{\Delta'}\rangle.$$

By compactness of $\mathbf{S}_{\overline{y}}(M)$ [proposition 1.18], there are finite subsets $\widetilde{W}_1\subseteq W_1$ and $\widetilde{W}_2\subseteq W_2$ such that

$$\mathbf{S}^{\mathfrak{C}}_{\overline{y}}(M)=\bigcup_{\Delta\in\widetilde{W}_1}\langle\chi_\Delta\rangle\cup\bigcup_{\Delta'\in\widetilde{W}_2}\langle\chi'_{\Delta'}\rangle.$$

Now, $\langle\bigvee_{\Delta\in\widetilde{W}_1}\chi_\Delta\rangle\subseteq\{\mathrm{tp}(\overline{b}/M)\,:\,\overline{b}\in\mathbf{X}\}$ and $\langle\bigvee_{\Delta\in\widetilde{W}_2}\chi_\Delta\rangle\subseteq\{\mathrm{tp}(\overline{b}/M)\,:\,\overline{b}\notin\mathbf{X}\}$, so

$$\mathbf{X}=\bigvee_{\Delta\in\widetilde{W}_1}\chi_\Delta[\mathfrak{C}].$$

Note that $\underline{\mathbf{X}}$ is an M-formula and remember that MR is invariant for elementary substructures, so $X=\underline{\mathbf{X}}[\mathfrak{M}]$. Finally, to show that $\mathbf{X}$ is definable with the parameters of $\mathbf{D}$, it suffices to apply the theorem 1.29 in the monster model. □

Corollary 2.29. ***(Definability of types with Morley's rank)*** *Let $A\subseteq M$ be a sorted subset and $p(\overline{x})\in\mathbf{S}^{\mathfrak{M}}(A)$ a type with Morley's rank. Then, p is definable over A in the sense of types, i.e., for every $\varphi(\overline{x},\overline{y})\in\mathrm{For}L$ there is $\mathrm{d}_p\overline{x}\varphi(\overline{y})\in\mathrm{For}L(A)$ such that, for any $\overline{a}$ from A,*

$$\varphi(\overline{x},\overline{a})\in p\Leftrightarrow\mathfrak{M}\models\mathrm{d}_p\overline{x}\varphi(\overline{y})[\overline{a}/\overline{y}].$$

Moreover, let $\phi \in p$ be such that $\mathrm{MR}(p) = \mathrm{MR}(\phi)$ *and* $\mathrm{Md}(p) = \mathrm{Md}(\phi)$*, and* A_0 *the finite sorted set of parameters of* ϕ*. Then,* p *is* A_0*-definable.*

Proof. Let $\phi \in p$ be such that $\mathrm{MR}(p) = \mathrm{MR}(\phi) = \alpha$ and $\mathrm{Md}(p) = \mathrm{Md}(\phi)$. Write by proposition 2.20

$$p = \{\varphi \in \mathrm{For}_{\overline{x}}\, L(A) \ : \ \mathrm{MR}\left(\phi[\mathfrak{M}] \setminus \varphi[\mathfrak{M}]\right) < \alpha\}.$$

Thus, for any $\varphi(\overline{x}, \overline{y})$, $\mathrm{d}_p \overline{x} \varphi$ is the A_0-formula defining the set $\{\overline{a} \ : \ \mathrm{MR}(\phi[\mathfrak{M}] \setminus \varphi[\mathfrak{M}]) < \alpha\}$, given by theorem 2.28. □

Note that this result is also true for global types in monster models. Also, note that a global type $\mathbf{p}$ with Morley's rank is definable over a sorted subset A if and only if $\mathrm{cb}(\mathbf{p}) \subseteq \mathrm{dcl}^{\mathrm{eq}}(A)$. Indeed, if $\mathbf{p}$ is definable over A, any automorphism fixing A leaves $\mathbf{p}$ invariant, so fixes $\mathrm{cb}(\mathbf{p})$. On the other hand, if $\mathrm{cb}(\mathbf{p}) \subseteq \mathrm{dcl}^{\mathrm{eq}}(A)$, any automorphism fixing A fixes $\mathrm{cb}(\mathbf{p})$, so it leaves $\mathbf{p}$ invariant and leaves $\mathrm{d}_{\mathbf{p}} \overline{x} \varphi[\mathfrak{C}]$ invariant, for every $\varphi \in \mathrm{For}_{\overline{x}, \overline{y}} L$. Therefore, every $\mathrm{d}_{\mathbf{p}} \overline{x} \varphi[\mathfrak{C}]$ is A-definable.

Finally, note that the corollary implies that the set $\{\mathrm{cb}(\mathrm{d}_{\mathbf{p}} \overline{x} \phi(\overline{y})) \ : \ \phi \in \mathrm{For}\, L\}$ is a canonical base of the global type $\mathbf{p}$. Thus, every global type with Morley's rank has a canonical base.

Corollary 2.30. *Let* $\mathfrak{C}$ *be the monster model of a totally transcendental* L*-theory,* $A \subseteq \mathbf{C}$ *a sorted subset and* $\mathbf{D}$ *an* A*-definable class. Then, every definable subclass of* $\mathbf{D}$ *is definable with parameters from* $A \cup \mathbf{D}$*.*

Proof. Let $\psi(\overline{x}, \overline{c})[\mathfrak{C}] \subseteq \mathbf{D}$. Then, by the corollary 2.29, we conclude that

$$\psi(\overline{x}, \overline{c})[\mathfrak{C}] = \{\overline{a} \in \mathbf{D} \ : \ \psi(\overline{a}, \overline{y}) \in \mathrm{tp}(\overline{c}/\mathbf{D})\} = \mathbf{D} \cap \mathrm{d}_{\mathrm{tp}(\overline{c}/\mathbf{D})} \overline{x} \psi[\mathfrak{C}].$$

Therefore, $\psi(\overline{x}, \overline{c})[\mathfrak{C}]$ is definable with the parameters of $\mathbf{D}$ and ϕ. Hence, it is $A \cup \mathbf{D}$-definable. □

2.4 Forking and independence

Let $A \subseteq M$ and $B \subseteq M$ be sorted subsets such that $A \subseteq B$ and $p \in \mathbf{S}^{\mathfrak{M}}(B)$ a type with Morley's rank. We say that p *forks* over A when $\mathrm{MR}(p) < \mathrm{MR}(p_{|A})$ and does not fork if $\mathrm{MR}(p) = \mathrm{MR}(p_{|A})$. Let $q \in \mathbf{S}^{\mathfrak{M}}(A)$, p is a *non-fork extension* of q when p does not forks over A and $q \subseteq p$, i.e., $p_{|A} = q$. We say that p is *stationary* when there is just one global non-forking extension in an $\aleph_0$-saturated elementary extension. For a stationary type p, write $\mathrm{cb}(p) := \mathrm{cb}(q)$ where q is the global non-forking extension of p.

Note that, for any $p \in \mathbf{S}^{\mathfrak{M}}_{\overline{x}}(A)$ with Morley's rank and any $B \supseteq A$, there is a non-forking extension of p to B. Indeed, by proposition 2.20,

$$p = \{\psi \in \mathrm{For}\, L(A) \ : \ \mathrm{MR}(\phi \wedge \neg\psi) < \mathrm{MR}(p)\}$$

with $\phi \in p$ such that $\mathrm{MR}(p) = \mathrm{MR}(\phi)$ and $\mathrm{Md}(p) = \mathrm{Md}(\phi)$. Then, apply propositions 2.12 and 2.21. The same is also true for a sorted class in a monster model. Moreover, we know that

$$\mathrm{Md}(p) = \sum \left\{ \mathrm{Md}(q) \,:\, q \in \mathbf{S}^{\mathfrak{M}}(B) \text{ non-forking extension of } p \right\}.$$

Consequently, we know that a type is stationary if and only if it has Morley's degree 1.

Proposition 2.31.

> (1) **Transitivity and Monotonicity:** Let $A, B, C \subseteq M$ be sorted sets such that $A \subseteq B \subseteq C$ and $p \in \mathbf{S}^{\mathfrak{M}}(A)$, $q \in \mathbf{S}^{\mathfrak{M}}(B)$ and $r \in \mathbf{S}^{\mathfrak{M}}(C)$ types such that $p \subseteq q \subseteq r$ and $\mathrm{MR}(p) \in \mathbb{O}\mathrm{n}$. Then, $p \subseteq r$ is a non-forking extension if and only if $p \subseteq q$ and $q \subseteq r$ are non-forking extensions.
>
> (2) **Finiteness:** Let $A \subseteq M$ and $p \in \mathbf{S}^{\mathfrak{M}}(A)$ with $\mathrm{MR}(p) \in \mathbb{O}\mathrm{n}$. Then, there is a finite sorted subset $A_0 \subseteq A$ such that p does not fork over A_0. Moreover, there is a finite $A_0 \subseteq A$ such that p is the unique non-forking extension of $p_{|A_0}$.

Proof.
(1) This properties are clear since $\mathrm{MR}(p) \geq \mathrm{MR}(q) \geq \mathrm{MR}(r)$ by definition.
(2) Let $\phi \in p$ be such $\mathrm{MR}(\phi) = \mathrm{MR}(p)$ and $\mathrm{Md}(\phi) = \mathrm{Md}(p)$ and A_0 the parameters of ϕ. □

Let $A, B \subseteq M$ and $\overline{a}$ from $\mathfrak{M}$ be such that $\overline{a}$ has Morley's rank over A. We say that $\overline{a}$ is *independent* from B over A if $\mathrm{tp}(\overline{a}/A \cup B)$ does not fork over A. Let A, B, C be sorted subsets such that every tuple from A has Morley's rank over C, we say that A is *independent* from B over C if every finite tuple from A is independent from B over C. We write $\overline{a} \mathop{\smile\!\!\!\!\!\!\;|}_A B$ and $A \mathop{\smile\!\!\!\!\!\!\;|}_C B$.

Remark. With this notation, the last properties of forking could be re-written in a more visual way. Let $A \subseteq M$ be a sorted subset and $\overline{a}$ a tuple from $\mathfrak{M}$ such that $\mathrm{MR}(\overline{a}/A) \in \mathbb{O}\mathrm{n}$.

> **Transitivity and Monotonicity:** Let $B, C \subseteq M$ be such that $A \subseteq B \subseteq C$. Then,
>
> $$\overline{a} \mathop{\smile\!\!\!\!\!\!\;|}_{A} B \cup C \Leftrightarrow \overline{a} \mathop{\smile\!\!\!\!\!\!\;|}_{A \cup B} C \text{ and } \overline{a} \mathop{\smile\!\!\!\!\!\!\;|}_{A} B.$$
>
> **Finiteness:** There is a finite $A_0 \subseteq A$ such that $\overline{a} \mathop{\smile\!\!\!\!\!\!\;|}_{A_0} A$. Moreover, by transitivity, there is a finite $A_0 \subseteq A$ such that, for any sorted subset B,
>
> $$\overline{a} \mathop{\smile\!\!\!\!\!\!\;|}_{A} B \Leftrightarrow \overline{a} \mathop{\smile\!\!\!\!\!\!\;|}_{A_0} B.$$

Lemma 2.32. *Let $\overline{a}, \overline{b}, \overline{a}', \overline{b}'$ be from $\mathfrak{M}$ such that $\mathrm{MR}(\overline{a}/A) \in \mathbb{O}\mathrm{n}$ and $A \subseteq M$. Then, $\mathrm{tp}(\overline{a}, \overline{b}/A) = \mathrm{tp}(\overline{a}', \overline{b}'/A)$ implies $\mathrm{MR}(\overline{a}/A, \overline{b}) = \mathrm{MR}(\overline{a}'/A, \overline{b}')$.*

Proof. Let $\mathfrak{C}$ be the monster extension of $\mathfrak{M}$. We prove the lemma by contradiction. Suppose that $\mathrm{MR}(\overline{a}/A,\overline{b}) < \mathrm{MR}(\overline{a}'/A,\overline{b}') = \alpha \in \mathbb{O}\mathrm{n}$. Let $\varphi \in \mathrm{tp}(\overline{a},\overline{b}/A)$ be such that $\mathrm{MR}(\varphi(\overline{x},\overline{b}')) < \alpha = \mathrm{MR}(\varphi(\overline{x},\overline{b}))$. Let $\psi \in \mathrm{tp}(\overline{a}/A)$ be such that $\mathrm{MR}(\psi) = \mathrm{MR}(\overline{a}/A) \geq \alpha$. By the theorem 2.28, let

$$\phi[\mathfrak{C}] = \{\overline{d} \,:\, \mathrm{MR}(\psi \wedge \varphi(\overline{x},\overline{d})) < \alpha\}.$$

Now, $\mathfrak{C} \models \phi[\overline{b}']$ and $\mathfrak{C} \not\models \phi[\overline{b}]$ where $\phi \in \mathrm{For}\,L(A)$, a contradiction since $\mathrm{tp}(\overline{b}'/A) = \mathrm{tp}(\overline{b}/A)$. $\square$

Theorem 2.33. ***(Symmetry)*** *Let $\mathfrak{N}$ be an infinite L-structure, $A \subseteq N$ a sorted set and $\overline{a}$ and $\overline{b}$ tuples with Morley's rank over A. Then, $\overline{a} \mathop{\smile\!\!\!\!\!\!\mid}_A \overline{b}$ if and only if $\overline{b} \mathop{\smile\!\!\!\!\!\!\mid}_A \overline{a}$.*

Proof. Let $\mathfrak{C}$ be the monster extension of $\mathfrak{N}$. It suffices to prove that $\overline{a} \mathop{\smile\!\!\!\!\!\!\mid}_A \overline{b}$ implies $\overline{b} \mathop{\smile\!\!\!\!\!\!\mid}_A \overline{a}$. Let $\mathfrak{M}$ be an $\aleph_0$-saturated elementary substructure such that $A \subseteq M$. Let $\overline{b}'$ from $\mathfrak{C}$ be such that $\mathrm{tp}(\overline{b}/A) \subseteq \mathrm{tp}(\overline{b}'/M)$ is a non-forking extension. Let $\overline{c}$ be from $\mathfrak{C}$ such that $\mathrm{tp}(\overline{a},\overline{b}/A) = \mathrm{tp}(\overline{c},\overline{b}'/A)$ and let $\overline{a}'$ be from $\mathfrak{C}$ such that $\mathrm{tp}(\overline{c}/A,\overline{b}') \subseteq \mathrm{tp}(\overline{a}'/M,\overline{b}')$ is a non-forking extension. Then, since $\mathrm{tp}(\overline{a},\overline{b}/A) = \mathrm{tp}(\overline{a}',\overline{b}'/A)$, by the lemma 2.32, we have that

$$\begin{aligned}&\mathrm{MR}(\overline{a}/A,\overline{b}) = \mathrm{MR}(\overline{a}'/A,\overline{b}') = \mathrm{MR}(\overline{a}'/M,\overline{b}'),\\ &\mathrm{MR}(\overline{b}/A,\overline{a}) = \mathrm{MR}(\overline{b}'/A,\overline{a}').\end{aligned}$$

So, $\overline{a} \mathop{\smile\!\!\!\!\!\!\mid}_A \overline{b}$ implies that $\overline{a}' \mathop{\smile\!\!\!\!\!\!\mid}_M \overline{b}'$ and $\overline{b}' \mathop{\smile\!\!\!\!\!\!\mid}_M \overline{a}'$ implies

$$\begin{aligned}\mathrm{MR}(\overline{b}/A,\overline{a}) =&\mathrm{MR}(\overline{b}'/A,\overline{a}') \geq \mathrm{MR}(\overline{b}'/M,\overline{a}') =\\ &=\mathrm{MR}(\overline{b}'/M) = \mathrm{MR}(\overline{b}/A) \geq \mathrm{MR}(\overline{b}/A,\overline{a}),\end{aligned}$$

i.e., $\overline{b} \mathop{\smile\!\!\!\!\!\!\mid}_A \overline{a}$. So it suffices to prove that $\overline{a}' \mathop{\smile\!\!\!\!\!\!\mid}_M \overline{b}'$ implies $\overline{b}' \mathop{\smile\!\!\!\!\!\!\mid}_M \overline{a}'$. Let $\varphi \in \mathrm{tp}_{\overline{x}}(\overline{a}'/M)$ and $\psi \in \mathrm{tp}_{\overline{y}}(\overline{b}'/M)$ be such that $\alpha = \mathrm{MR}(\varphi) = \mathrm{MR}(\overline{a}'/M)$, $\beta = \mathrm{MR}(\psi) = \mathrm{MR}(\overline{b}/M)$, $\mathrm{Md}(\varphi) = \mathrm{Md}(\overline{a}'/M)$ and $\mathrm{Md}(\psi) = \mathrm{Md}(\overline{b}'/M)$. If $\overline{b}' \not\mathop{\smile\!\!\!\!\!\!\mid}_M \overline{a}'$. There is $\chi' \in \mathrm{tp}_{\overline{x},\overline{y}}(\overline{a}',\overline{b}'/M)$ such that $\mathrm{MR}(\chi'(\overline{a}',\overline{y})) = \mathrm{MR}(\overline{b}'/M,\overline{a}') < \beta$. Consider $\chi = \chi' \wedge \varphi \wedge \psi$. By the theorem 2.28, let

$$\phi[\mathfrak{C}] = \{\overline{d} \,:\, \mathrm{MR}(\chi(\overline{d},\overline{y})) < \beta\} = \{\overline{d} \,:\, \mathrm{MR}(\psi \wedge \chi(\overline{d},\overline{y})) < \beta\}.$$

Then, $\chi(\overline{x},\overline{y}) \wedge \phi(\overline{x}) \in \mathrm{tp}(\overline{a}',\overline{b}'/M)$, so $\chi(\overline{x},\overline{b}') \wedge \phi(\overline{x}) \in \mathrm{tp}(\overline{a}'/M,\overline{b}')$. Since $\overline{a}' \mathop{\smile\!\!\!\!\!\!\mid}_M \overline{b}'$, $\mathrm{MR}(\chi(\overline{x},\overline{b}') \wedge \phi(\overline{x})) = \mathrm{MR}(\overline{a}'/M,\overline{b}') = \mathrm{MR}(\overline{a}'/M) = \mathrm{MR}(\varphi)$ and $\chi(\overline{x},\overline{b}') \wedge \phi(\overline{x})[\mathfrak{C}] \subseteq \varphi[\mathfrak{C}]$. Since $\mathfrak{M}$ is $\aleph_0$-saturated, by the lemma 2.27, there is $\overline{a}_0'$ from $\mathfrak{M}$ such that $\mathfrak{C} \models \chi \wedge \phi[\overline{a}_0',\overline{b}']$. However, in that case, $\mathrm{MR}(\chi(\overline{a}_0',\overline{y})) < \beta$ and $\chi(\overline{a}_0',\overline{y}) \in \mathrm{tp}(\overline{b}'/M)$ where $\beta = \mathrm{MR}(\overline{b}'/M)$, a contradiction. $\square$

Corollary 2.34. *Let $A \subseteq M$ be a sorted subset and $p \in \mathbf{S}^{\mathfrak{M}}(\operatorname{acl}(A))$ such that $p_{|A}$ has Morley's rank. Then, p does not fork over A.*

Proof. Consider an $\aleph_0$-saturated elementary extension $\mathfrak{M}'$ realizing p and take $\overline{a}$, a realization of p. Thus, p does not fork over A if and only if $\overline{a} \mathop{\smile\!\!\!\!\!\!|}_A \operatorname{acl}(A)$. By finiteness [proposition 2.31], let $\overline{b}$ be a finite tuple of $\operatorname{acl}(A)$ such that $\overline{a} \mathop{\smile\!\!\!\!\!\!|}_A \operatorname{acl}(A)$ if and only if $\overline{a} \mathop{\smile\!\!\!\!\!\!|}_A \overline{b}$. By symmetry [theorem 2.33], $\overline{a} \mathop{\smile\!\!\!\!\!\!|}_A \overline{b}$ if and only if $\overline{b} \mathop{\smile\!\!\!\!\!\!|}_A \overline{a}$. And the last one is trivial since $0 \leq \mathrm{MR}(\overline{b}/A, \overline{a}) \leq \mathrm{MR}(\overline{b}/A) = 0$ because $\overline{b}$ is algebraic. $\square$

Corollary 2.35. *Let $A, B \subseteq M$ be sorted subsets and $\overline{a}, \overline{b}$ from $\mathfrak{M}$ such that $\mathrm{MR}(\overline{a}, \overline{b}/A) \in \mathbb{O}\mathrm{n}$. Then,*

$$\overline{a}, \overline{b} \mathop{\smile\!\!\!\!\!\!|}_{A} B \Leftrightarrow \overline{a} \mathop{\smile\!\!\!\!\!\!|}_{A} B \text{ and } \overline{b} \mathop{\smile\!\!\!\!\!\!|}_{A,\overline{a}} B.$$

Proof. Note that $\overline{a} \mathop{\smile\!\!\!\!\!\!|}_A B$ if and only if $\overline{a} \mathop{\smile\!\!\!\!\!\!|}_A \overline{b}$ for every $\overline{b}$ finite from $\mathbf{B}$. Now, for every $\overline{c}$ from B,

$$\overline{a}, \overline{b} \mathop{\smile\!\!\!\!\!\!|}_{A} \overline{c} \Leftrightarrow \overline{c} \mathop{\smile\!\!\!\!\!\!|}_{A} \overline{a}, \overline{b} \Leftrightarrow \overline{c} \mathop{\smile\!\!\!\!\!\!|}_{A} \overline{a} \text{ and } \overline{c} \mathop{\smile\!\!\!\!\!\!|}_{A,\overline{a}} \overline{b} \Leftrightarrow \overline{a} \mathop{\smile\!\!\!\!\!\!|}_{A} \overline{c} \text{ and } \overline{b} \mathop{\smile\!\!\!\!\!\!|}_{A,\overline{a}} \overline{c}.$$

So, the corollary follows. $\square$

Now, we are going to prove the theorem 2.37, which states a fundamental relation between forking and canonical bases.

***Lemma* 2.36.** *Let $\mathfrak{C}$ be the monster extension of $\mathfrak{M}$ and $p \in \mathbf{S}^{\mathfrak{C}}_{\overline{x}}(M)$ with Morley's rank. Then, p has a unique global extension which is definable over M.*

Proof. By corollary 2.29, we have

$$\mathbf{p} = \{\varphi(\overline{x}, \overline{b}) \,:\, \varphi \in \operatorname{For} L(M) \text{ and } \overline{b} \in \mathrm{d}_p\overline{x}\varphi[\mathfrak{C}]\}.$$

Of course, $p \subseteq \mathbf{p}$ and $\mathbf{p}$ is M-definable. Since $\mathrm{d}_p\overline{x}\varphi(\overline{y}) \wedge \mathrm{d}_p\overline{x}\varphi'(\overline{y}) = \mathrm{d}_p\overline{x}(\varphi \wedge \varphi')(\overline{y})$, it suffices to show that for any $\varphi \in \operatorname{For} L(M)$,

$$\mathfrak{C} \models \forall \overline{y}(\, \mathrm{d}_p\overline{x}\varphi(\overline{y}) \rightarrow \exists \overline{x}\varphi(\overline{x}, \overline{y})).$$

That is clear since $\mathfrak{M} \prec \mathfrak{C}$ and $\mathfrak{M}$ satisfy that. Finally, to prove that $\mathbf{p}$ is a global type, note that $\neg\, \mathrm{d}_p\overline{x}\varphi(\overline{y}) = \mathrm{d}_p\overline{x}\neg\varphi(\overline{y})$, so either $\varphi(\overline{x}, \overline{b}) \in \mathbf{p}$ or $\neg\varphi(\overline{x}, \overline{b}) \in \mathbf{p}$ for any $\varphi \in \operatorname{For} L(M)$ and $\overline{b}$ from $\mathbf{C}$. Let us prove that $\mathbf{p}$ is the unique global extension definable over M. Let $\mathbf{q}$ be another one. Then, for any $\varphi \in \operatorname{For} L(M)$, we have that $\mathrm{d}_{\mathbf{q}}\overline{x}\varphi[\mathfrak{M}] = \mathrm{d}_p\overline{x}\varphi[\mathfrak{M}] = \mathrm{d}_{\mathbf{p}}\overline{x}\varphi[\mathfrak{M}]$. Since $\mathfrak{M} \preceq \mathfrak{C}$, we conclude

$$\varphi(\overline{x}, \overline{b}) \in \mathbf{q} \Leftrightarrow \overline{b} \in \mathrm{d}_{\mathbf{q}}\overline{x}\varphi[\mathfrak{C}] = \mathrm{d}_{\mathbf{p}}\overline{x}\varphi[\mathfrak{C}] \Leftrightarrow \varphi(\overline{x}, \overline{b}) \in \mathbf{p}.$$

$\square$

Theorem 2.37. ***(Canonical base and forking)*** *Let $\mathfrak{C}$ be the monster model of a totally transcendental L-theory,* $\mathbf{p}$ *a global type and A a sorted subset. Then,* $\mathbf{p}$ *does not fork over A if and only if* $\mathbf{p}$ *is definable over* $\mathrm{acl}(A)$*, i.e.,* $\mathrm{cb}(\mathbf{p}) \subseteq \mathrm{acl}^{\mathrm{eq}}(A)$.

Proof. ($\Rightarrow$) By the proposition 2.20 we know that $p = \mathbf{p}_{|A} = \{\psi \in \mathrm{For}_{\bar{x}}L(A) : \mathrm{MR}(\phi \wedge \neg\psi) < \mathrm{MR}(p)\}$ where $\phi \in p$, $\mathrm{MR}(\phi) = \mathrm{MR}(p)$ and $\mathrm{Md}(\phi) = \mathrm{Md}(p)$. By the proposition 2.12, let $\phi = \phi^1 \vee \cdots \vee \phi^d$ be a partition of ϕ with $\mathrm{MR}(\phi^i) = \mathrm{MR}(\phi)$ and $\mathrm{Md}(\phi^i) = 1$ for each i. We may assume that $\phi^1 \in \mathbf{p}$. Then, $\mathbf{p} = \{\psi \in \mathrm{For}_{\bar{x}}L(\mathbf{C}) : \mathrm{MR}(\phi^1 \wedge \neg\psi) < \mathrm{MR}(p)\}$. An automorphism $\mathbf{f}$ which fixes A leaves ϕ invariant, so $\mathbf{f}(\mathbf{p})$ must have one of the ϕ^i. In that case, since $\mathrm{MR}(\phi^i) = \mathrm{MR}(\phi) = \mathrm{MR}(p) = \mathrm{MR}(\mathbf{p}) = \mathrm{MR}(\mathbf{f}(\mathbf{p}))$ and $\mathrm{Md}(\phi^i) = 1 = \mathrm{Md}(\mathbf{p}) = \mathrm{Md}(\mathbf{f}(\mathbf{p}))$, it is clear that $\mathbf{f}(\mathbf{p}) = \{\psi \in \mathrm{For}_{\bar{x}}L(\mathbf{C}) : \mathrm{MR}(\phi^i \wedge \neg\psi) < \mathrm{MR}(p)\}$. Therefore, there are at most d different conjugates of $\mathbf{p}$ over A. Since $\mathbf{p}$ is definable [corollary 2.29], by the theorem 1.29, $\mathbf{p}$ is definable over $\mathrm{acl}(A)$.

($\Leftarrow$) We want to prove that $\mathbf{p}_{|A} \subseteq \mathbf{p}_{|B}$ is a non-forking extension for any sorted subset B extending A. Let $\mathfrak{M}$ be an elementary substructure of $\mathfrak{C}$ such that $A \subseteq M$. We may assume that $M \mathop{\smile\!\!\!\!\!\!\mid}_A B$. Indeed, let $\{m_\xi\}_{\xi\in\alpha}$ be the disjoint union of a $\{M_s\}_{s\in S}$. We define the sorted set $\widetilde{M}$, whose elements are $\{\widetilde{m}_\xi\}_{\xi\in\alpha}$, by recursion such way that $\mathrm{tp}(\widetilde{m}_\xi/B \cup \{\widetilde{m}_\xi\}_{\eta\in\xi})$ does not fork over $A \cup \{\widetilde{m}_\xi\}_{\eta\in\xi}$ and $f : M \to \widetilde{M}$ given by $f : m_\xi \mapsto \widetilde{m}_\xi$ is an elementary map fixing A, for each $\xi \in \alpha$. To do that, choose an element realizing a non-forking extension to B of $\mathrm{tp}(m_0/A)$ as $\widetilde{m}_0$ and, given $\{\widetilde{m}_\eta\}_{\xi\in\eta}$ and $f_\xi : m_\eta \mapsto \widetilde{m}_\eta$, let g be an elementary map extending f_ξ to m_ξ, which exists by homogeneity of $\mathfrak{C}$, and let $\widetilde{m}_\xi$ be an element in $\mathfrak{C}$ realizing a non-forking extension to B of $\mathrm{tp}(g(m_\xi)/A \cup \{\widetilde{m}_\eta\}_{\eta\in\xi})$, which exists by saturation of $\mathfrak{C}$. Since $\mathrm{tp}(\widetilde{m}_\xi/A \cup \{\widetilde{m}_\eta\}) = \mathrm{tp}(g(m_\xi)/A \cup \{\widetilde{m}_\eta\}_{\eta\in\xi})$ and g is an elementary map, it is clear that $f_{\xi+1} : m_\eta \mapsto \widetilde{m}_\eta$ for $\eta \leq \xi$ is an elementary map. Now, since f is an elementary map fixing A, $\widetilde{M}$ defines an elementary substructure of $\mathfrak{C}$ by the Tarski's test [Theorem 1.7] such that $A \subseteq \widetilde{M}$. Also, note that $\widetilde{M} \mathop{\smile\!\!\!\!\!\!\mid}_A B$. Indeed, we prove that by induction. We want to show that $\{\widetilde{m}_\eta\}_{\eta\leq\xi} \mathop{\smile\!\!\!\!\!\!\mid}_A B$ when $\{\widetilde{m}_\eta\}_{\eta<\xi} \mathop{\smile\!\!\!\!\!\!\mid}_A B$. We know that

$$\widetilde{m}_\xi, \widetilde{m}_{\eta_1} \dots, \widetilde{m}_{\eta_n} \mathop{\smile\!\!\!\!\!\!\mid}_A B \Leftrightarrow \widetilde{m}_\xi \mathop{\smile\!\!\!\!\!\!\mid}_{A, \widetilde{m}_{\eta_1}, \dots, \widetilde{m}_{\eta_n}} B \text{ and } \widetilde{m}_{\eta_1}, \dots, \widetilde{m}_{\eta_n} \mathop{\smile\!\!\!\!\!\!\mid}_A B$$

by the corollary 2.35. But $\widetilde{m}_\xi \mathop{\smile\!\!\!\!\!\!\mid}_{A\cup\{\widetilde{m}_\eta\}_{\eta\in\xi}} B$ by construction of $\widetilde{M}$ and by induction hypothesis $\{\widetilde{m}_\eta\}_{\eta\in\xi} \mathop{\smile\!\!\!\!\!\!\mid}_A B$. So assume that M is already independent from B over A. Since $\mathbf{p}$ is definable over $\mathrm{acl}(A)$, it is definable over M. Also, $\mathbf{p}_{|M}$ is definable over M. Thus, $\mathbf{p}$ is the unique extension of $\mathbf{p}_{|M}$ which is definable over M [lemma 2.36]. On the other hand, a global non-forking extension of $\mathbf{p}_{|M}$ is definable over M since it does not fork over M [corollary 2.29]. Thus, $\mathbf{p}$ is the unique non-forking extension of $\mathbf{p}_{|M}$. Now, let $\overline{a}$ realize $\mathbf{p}_{|B\cup M}$. Then, $\overline{a} \mathop{\smile\!\!\!\!\!\!\mid}_M B$. Now, for any $\overline{b}$ from B, we have by finiteness, transitivity and monotonicity [proposition 2.31] that

$$\bar{a} \underset{M}{\perp} \bar{b} \Leftrightarrow \bar{a} \underset{M_0}{\perp} \bar{b} \text{ and } M_0 \underset{A}{\perp} \bar{b} \Rightarrow \bar{a} \underset{A}{\perp} \bar{b}, M_0 \Rightarrow \bar{a} \underset{A}{\perp} \bar{b},$$

where $M_0 \subseteq M$ is the finite sorted subset given by the property of finiteness and we use $M_0 \perp_A B$. Therefore, $\bar{a} \perp_A B$, so $\mathbf{p}_{|A} \subseteq \mathbf{p}_{|A\cup B}$ is a non-forking extension. □

Note that We have not used that the theory is totally transcendental to prove $(\Rightarrow)$. Remember that $\mathbf{p}$ is definable over $\mathrm{acl}(A)$ if and only if $\mathrm{cb}(\mathbf{p}) \subseteq \mathrm{acl}^{\mathrm{eq}}(A)$, noted just after the corollary 2.29.

Theorem 2.38. *Let $\mathfrak{C}$ be the monster model of a totally transcendental L-theory. Then, any strong type in $\mathfrak{C}^{\mathrm{eq}}$ is stationary.*

Proof. Let $A \subseteq \mathbf{C}^{\mathrm{eq}}$ be a sorted subset $\mathrm{acl}^{\mathrm{eq}}$-closed and $p \in \mathbf{S}_{\bar{x}}^{\mathfrak{C}^{\mathrm{eq}}}(A)$. Let $q_1, q_2 \in \mathbf{S}_{\bar{x}}^{\mathfrak{C}^{\mathrm{eq}}}(B)$ be two non-forking extensions of p to some $A \subseteq B$. Thus, by proposition 2.20, $q_1 = \{\psi \in \mathrm{For}_{\bar{x}} L^{\mathrm{eq}}(B) : \mathrm{MR}(\phi_1 \wedge \neg\psi) < \alpha\}$ and $q_2 = \{\psi \in \mathrm{For}_{\bar{x}} L^{\mathrm{eq}}(B) : \mathrm{MR}(\phi_2 \wedge \neg\psi) < \alpha\}$ where $\alpha = \mathrm{MR}(p) = \mathrm{MR}(q_1) = \mathrm{MR}(q_2)$, $\mathrm{Md}_\alpha(\phi_1) = \mathrm{Md}_\alpha(q_1)$ and $\mathrm{Md}_\alpha(\phi_2) = \mathrm{Md}_\alpha(q_2)$. Note that if $\phi_1 \in q_2$ and $\phi_2 \in q_1$, then $q_1 = q_2$. So, let $\bar{b}$ be the finite tuple of parameters of ϕ_1 and ϕ_2, then $q_1 = q_2$ if and only if $q_{1|A,\bar{b}} = q_{2|A,\bar{b}}$. Let $\bar{a}_1$ realize $q_{1|A,\bar{b}}$ and $\bar{a}_2$ realize a non-forking extension of $q_{2|A,\bar{b}}$ to $A, \bar{b}, \bar{a}_1$, so $\bar{a}_2 \perp_{A,\bar{b}} \bar{a}_1$. Since $p \subseteq q_1$ and $p \subseteq q_2$ are non-forking extensions, $\bar{a}_1 \perp_A \bar{b}$ and $\bar{a}_2 \perp_A \bar{b}$. Then, by transitivity and monotonicity, $\bar{a}_2 \perp_{A,\bar{a}_1} \bar{b}$ [proposition 2.31]. By the corollary 2.35, $\bar{a}_1, \bar{a}_2 \perp_A \bar{b}$. By symmetry [theorem 2.33], we conclude that $r = \mathrm{tp}(\bar{b}/A, \bar{a}_1, \bar{a}_2)$ does not fork over A. Let $\mathbf{r}$ be a global non-forking extension of r. By the theorem 2.37, since $A \subseteq \mathrm{dcl}^{\mathrm{eq}}(A) \subseteq \mathrm{acl}^{\mathrm{eq}}(A) = A$, we conclude that, for every $\varphi \in \mathrm{For}_{\bar{x},\bar{y}} L^{\mathrm{eq}}(A)$, there is a formula $\mathrm{d}_{\mathbf{r}}\bar{y}\varphi(\bar{x}) \in \mathrm{For}_{\bar{x}} L^{\mathrm{eq}}(A)$ such that

$$\begin{aligned}\varphi(\bar{a}_1, \bar{y}) \in r \subseteq \mathbf{r} \Leftrightarrow & \mathfrak{C} \models \mathrm{d}_{\mathbf{r}}\bar{y}\varphi[\bar{a}_1/\bar{x}] \Leftrightarrow \\ & \Leftrightarrow \mathrm{d}_{\mathbf{r}}\bar{y}\varphi(\bar{x}) \in \mathrm{tp}(\bar{a}_1/A) = p = \mathrm{tp}(\bar{a}_2/A) \Leftrightarrow \\ & \Leftrightarrow \mathfrak{C}^{\mathrm{eq}} \models \mathrm{d}_{\mathbf{r}}\bar{y}\varphi[\bar{a}_2/\bar{x}] \Leftrightarrow \varphi(\bar{a}_2, \bar{y}) \in r \subseteq \mathbf{r}.\end{aligned}$$

Thus, $\mathfrak{C}^{\mathrm{eq}} \models \varphi[\bar{a}_1, \bar{b}] \Leftrightarrow \mathfrak{C}^{\mathrm{eq}} \models \varphi[\bar{a}_2, \bar{b}]$ for every $\varphi \in \mathrm{For}_{\bar{x},\bar{y}} L^{\mathrm{eq}}(A)$. Hence, $q_{1|A,\bar{b}} = q_{2|A,\bar{b}}$, so $q_1 = q_2$. □

Corollary 2.39. *Let $\mathfrak{C}$ be the monster model of totally transcendental L-theory, A a sorted subset and $p \in \mathbf{S}_{\bar{x}}^{\mathfrak{C}}(A)$ with Morley's rank. Then, all the global non-forking extensions of p are conjugate over A.*

Proof. Let $\mathbf{q}_1, \mathbf{q}_2$ be two global non-forking extensions of p. Then, $q_1^{\mathrm{eq}} = {\mathbf{q}_1^{\mathrm{eq}}}_{|\mathrm{acl}^{\mathrm{eq}}(A)}$ and $q_2^{\mathrm{eq}} = {\mathbf{q}_2^{\mathrm{eq}}}_{|\mathrm{acl}^{\mathrm{eq}}(A)}$ are two stationary non-forking extensions of p^{eq} [theorem 2.38]. Let $\bar{a}_1$ and $\bar{a}_2$ realize q_1^{eq} and q_2^{eq}. Since $\mathrm{tp}(\bar{a}_1/A) = \mathrm{tp}(\bar{a}_2/A)$, by the lemma 1.28, there is an automorphism $\mathbf{f}$ fixing A such that $\mathbf{f}(\bar{a}_1) = \bar{a}_2$. Consider $\mathbf{f}^{\mathrm{eq}}$ the natural extension of $\mathbf{f}$ to $\mathfrak{C}^{\mathrm{eq}}$. Then, $\mathbf{f}^{\mathrm{eq}}(q_1^{\mathrm{eq}}) = q_2^{\mathrm{eq}}$. Now,

$\mathrm{MR}(\mathbf{f}^{\mathrm{eq}}(\mathbf{q}_1^{\mathrm{eq}})) = \mathrm{MR}(\mathbf{q}_1^{\mathrm{eq}}) = \mathrm{MR}(p^{\mathrm{eq}}) = \mathrm{MR}(q_2^{\mathrm{eq}}) = \mathrm{MR}(q_2)$, so $\mathbf{f}^{\mathrm{eq}}(\mathbf{q}_1^{\mathrm{eq}})$ is the global non-forking extension of q_2^{eq}, which is stationary by the theorem 2.38. Hence, $\mathbf{f}^{\mathrm{eq}}(\mathbf{q}_1^{\mathrm{eq}}) = \mathbf{q}_2^{\mathrm{eq}}$, so $\mathbf{f}(\mathbf{q}_1) = \mathbf{q}_2$. $\square$

Corollary 2.40. *Let $\mathfrak{C}$ be the monster model of a totally transcendental L-theory, $\mathbf{p}$ a global type and A a sorted subset. Then, $\mathbf{p}$ does not fork over A and $\mathbf{p}_{|A}$ is stationary if and only if* $\mathrm{cb}(\mathbf{p}) \subseteq \mathrm{dcl}^{\mathrm{eq}}(A)$.

Proof. Since $\mathbf{p}$ does not fork over A if and only if $\mathrm{cb}(\mathbf{p}) \in \mathrm{acl}^{\mathrm{eq}}(A)$ [theorem 2.37], it suffices to prove that, when $\mathbf{p}$ does not fork over A, $\mathbf{p}_{|A}$ is stationary if and only if $\mathrm{cb}(\mathbf{p}) \in \mathrm{dcl}^{\mathrm{eq}}(A)$. By the corollary 2.39, $\mathbf{p}$ is the unique non-forking extension of $\mathbf{p}_{|A}$ if and only if $\mathbf{p}$ is invariant for automorphism fixing A. Therefore, $\mathrm{cb}(\mathbf{p}) \subseteq \mathrm{dcl}^{\mathrm{eq}}(A)$. $\square$

Corollary 2.41. ***(Finiteness of canonical bases)*** *Let $\mathfrak{C}$ be the monster model of a totally transcendental L-theory and $\mathbf{p}$ a global type with Morley's rank. Then, there is a $\phi \in \mathbf{p}$ such that $\mathrm{cb}(\phi)$ is a canonical base of $\mathbf{p}$.*

Proof. Let $\mathrm{cb}(\mathbf{p})$ be a canonical base of $\mathbf{p}$. Since $\mathrm{cb}(\mathbf{p}) \subseteq \mathrm{dcl}^{\mathrm{eq}}(\mathrm{cb}(\mathbf{p}))$, $\mathbf{p}$ is the unique non-forking extension of $\mathbf{p}_{|\mathrm{cb}(\mathbf{p})}$. Then, $\mathbf{p}_{|\mathrm{cb}(\mathbf{p})} = \{\psi \in \mathrm{For}_{\overline{x}}L(\mathrm{cb}(\mathbf{p})) : \mathrm{MR}(\phi \wedge \neg\psi) < \mathrm{MR}(\phi)\}$ where $\phi \in \mathbf{p}_{|\mathrm{cb}(\mathbf{p})}$ is such that $\mathrm{MR}(\phi) = \mathrm{MR}(\mathbf{p})$ and $\mathrm{Md}(\phi) = 1$ — we may assume that $\phi \in \mathrm{For}_{\overline{x}}L(\mathrm{cb}(\mathbf{p}))$ and not in $\mathrm{For}L^{\mathrm{eq}}(\mathrm{cb}(\mathbf{p}))$. Thus, $\mathrm{cb}(\phi) \in \mathrm{dcl}^{\mathrm{eq}}(\mathrm{cb}(\mathbf{p}))$. On the other hand, $\mathrm{MR}(\mathbf{p}_{|\mathrm{cb}(\phi)}) = \mathrm{MR}(\phi) = \mathrm{MR}(\mathbf{p})$ and $\mathrm{Md}(\mathbf{p}_{|\mathrm{cb}(\phi)}) = \mathrm{Md}(\phi) = 1$. That implies that $\mathbf{p}$ is the unique non-forking extension of $\mathbf{p}_{|\mathrm{cb}(\phi)}$, so $\mathrm{cb}(\mathbf{p}) \in \mathrm{dcl}^{\mathrm{eq}}(\mathrm{cb}(\phi))$. Hence, $\mathrm{cb}(\phi)$ is a canonical base of $\mathbf{p}$. $\square$

Let $\mathfrak{C}$ be the monster extension of $\mathfrak{M}$, $p \in \mathbf{S}^{\mathfrak{C}}_{\overline{x}}(M)$, B a sorted set such that $M \subseteq B$ and $q \in \mathbf{S}^{\mathfrak{C}}_{\overline{x}}(B)$ such that $p \subseteq q$. We say that q is a *heir* of p if, for every tuple $\overline{b}$ and every M-formula φ such that $\varphi(\overline{x}, \overline{b}) \in q$, there is a tuple $\overline{m}$ from M such that $\varphi(\overline{x}, \overline{m}) \in p$. We say that q is a *coheir* of p if, for every $\varphi(\overline{x}, \overline{b}) \in q$, there is a tuple $\overline{m}$ from M which realizes $\varphi(\overline{x}, \overline{b})$.

By a straightforward use of the Zorn's lemma, we can prove there are always heirs and coheirs for any complete theory whose models are infinite.

Theorem 2.42. ***(Heirs and Coheirs)*** *Let T be a totally transcendental L-theory, $\mathfrak{M} \models T$, $\mathfrak{C}$ the monster model of T, $p \in \mathbf{S}^{\mathfrak{C}}_{\overline{x}}(M)$, B a sorted set such that $M \subseteq B$ and $q \in \mathbf{S}^{\mathfrak{C}}_{\overline{x}}(B)$ such that $p \subseteq q$. Then, the following are equivalent:*
(1) q is a heir of p.
(2) q is a coheir of p.
(3) q is the non-forking extension of p.

Proof. (3)$\Rightarrow$(1) By the theorem 2.37, q is definable over $M = \mathrm{acl}(M)$. So, for every M-formula $\varphi(\overline{x}, \overline{y})$ and every tuple $\overline{b}$, there is an M-formula $\mathrm{d}_q\overline{x}\varphi(\overline{y})$ such that

$$\begin{aligned}\varphi(\overline{x}, \overline{b}) \in q \Leftrightarrow & \mathfrak{C} \models \mathrm{d}_q\overline{x}\varphi[\overline{b}] \Rightarrow \mathfrak{C} \models \exists \overline{y}\, \mathrm{d}_q\overline{x}\varphi(\overline{y}) \Leftrightarrow \\ & \Leftrightarrow \mathfrak{M} \models \overline{y}\, \mathrm{d}_q\overline{x}\varphi(\overline{y}) \Leftrightarrow \text{ there is } \overline{m} \text{ from } M \quad \mathfrak{C} \models \mathrm{d}_q\overline{x}\varphi[\overline{m}] \Leftrightarrow \\ & \Leftrightarrow \text{ there is } \overline{m} \text{ from } M \quad \varphi(\overline{x}, \overline{m}) \in q.\end{aligned}$$

So q is a heir of p.
(1)$\Rightarrow$(3) By the theorem 2.37 and the lemma 2.36, it is clear that p is stationary. Indeed, given two global non-forking extensions of p, both are M-definable, so these ones are equal. Suppose that q is not a non-forking extension of p and let $\mathbf{p}$ be the global non-forking extension of p and $q' = \mathbf{p}_{|B}$. Let $\neg\phi(\overline{x}, \overline{b}) \in q$ be such that $\phi(\overline{x}, \overline{b}) \in q'$. Then, we have that $\mathfrak{C} \models \mathrm{d}_{\mathbf{p}}\overline{x}\phi[\overline{b}]$ with $\mathrm{d}_{\mathbf{p}}\overline{x}\phi \in \mathrm{For}_{\overline{y}}L(M)$. So $\mathrm{d}_{\mathbf{p}}\overline{x}\phi(\overline{b}) \in q$. But $\neg\phi(\overline{x}, \overline{b}) \in q$, so $\neg\phi(\overline{x}, \overline{b}) \wedge \mathrm{d}_{\mathbf{p}}\overline{x}\phi(\overline{b}) \in q$. Since q is a heir of p, there is $\overline{m}$ from M such that $\neg\phi(\overline{x}, \overline{m}) \wedge \mathrm{d}_{\mathbf{p}}\overline{x}\phi(\overline{m}) \in p$. Then, $\neg\phi(\overline{x}, \overline{m}) \in p$ and $\phi(\overline{x}, \overline{m}) \in p$, a contradiction.
(1)$\Rightarrow$(2) Let $\overline{a}$ be such that $q = \mathrm{tp}(\overline{a}/B)$. Assume that $\mathrm{tp}(\overline{a}/B)$ is a heir of $\mathrm{tp}(\overline{a}/M)$. Let $\overline{b}$ be a finite tuple from B. Let $\phi(\overline{x}, \overline{b}) \in q$, we want to prove that $\phi(\overline{x}, \overline{b})$ is realized in M. Since (1)$\Rightarrow$(3), we know that $\overline{a} \mathop{\smile\!\!\!\!\!\mid}_M \overline{b}$. By symmetry [theorem 2.33], $\overline{b} \mathop{\smile\!\!\!\!\!\mid}_M \overline{a}$. Therefore, since (3)$\Rightarrow$(1), $\mathrm{tp}(\overline{b}/M, \overline{a})$ is a heir of $\mathrm{tp}(\overline{b}/M)$. Now, $\phi(\overline{a}, \overline{y}) \in \mathrm{tp}(\overline{b}/M, \overline{a})$, so there is an $\overline{m}$ from M such that $\phi(\overline{m}, \overline{y}) \in \mathrm{tp}(\overline{b}, M)$. So, there is an $\overline{m}$ from M which realizes $\phi(\overline{x}, \overline{b})$.
(2)$\Rightarrow$(1) Let $\overline{a}$ be such that $q = \mathrm{tp}(\overline{a}/B)$. Assume that $\mathrm{tp}(\overline{a}/B)$ is a coheir of $\mathrm{tp}(\overline{a}/M)$. Let $\overline{b}$ be a finite tuple from B. Consider $\mathrm{tp}(\overline{b}/M, \overline{a})$. Then, for every $\phi(\overline{a}, \overline{y}) \in \mathrm{tp}(\overline{b}/M, \overline{a})$, there is an $\overline{m}$ from M such that $\mathfrak{C} \models \phi(\overline{x}, \overline{b})[\overline{m}]$. So, there is an $\overline{m}$ from M such that $\phi(\overline{m}, \overline{y}) \in \mathrm{tp}(\overline{b}/M)$. Thus, $\mathrm{tp}(\overline{b}/M, \overline{a})$ is a heir of $\mathrm{tp}(\overline{b}/M)$. Since (1)$\Rightarrow$(2), $\mathrm{tp}(\overline{b}/M, \overline{a})$ is a coheir of $\mathrm{tp}(\overline{b}/M)$. Let $\phi(\overline{x}, \overline{b}) \in \mathrm{tp}(\overline{a}/M, \overline{b})$, then $\phi(\overline{a}, \overline{y}) \in \mathrm{tp}(\overline{b}/M, \overline{b})$. Hence, there is an $\overline{m}$ from M realizing $\phi(\overline{a}, \overline{y})$, i.e., there is an $\overline{m}$ from M such that $\phi(\overline{x}, \overline{m}) \in \mathrm{tp}(\overline{a}/M)$. □

Theorem 2.43. *(Closedness) Let $\mathfrak{C}$ be the monster model of a totally transcendental L-theory, A a sorted subsets and $\mathbf{B}$ be a sorted subclass such that $A \subseteq \mathbf{B}$. Then, the set $\mathrm{NF}(\mathbf{B}/A) \subseteq \mathbf{S}^{\mathfrak{C}}_{\overline{x}}(\mathbf{B})$ of types which does not fork over A is closed.*

Proof. Let $\mathbf{q} \in \mathbf{S}^{\mathfrak{C}}_{\overline{x}}(\mathbf{B}) \setminus \mathrm{NF}(\mathbf{B}/A)$. Since $\mathbf{q}$ forks, there are $\overline{b}$ finite from B and $\overline{a}$ such that $\mathrm{tp}(\overline{a}/A, \overline{b}) = \mathbf{q}_{|A,\overline{b}}$ and $\overline{a} \not\mathop{\smile\!\!\!\!\!\mid}_A \overline{b}$. By symmetry [Theorem 2.33], $\overline{b} \not\mathop{\smile\!\!\!\!\!\mid}_A \overline{a}$. Let $\mathbf{r}_1, \ldots, \mathbf{r}_d$ be the global non-forking extensions of $\mathrm{tp}(\overline{a}/A)$. Since $\mathrm{tp}(\overline{b}/A, \overline{a}) \neq \mathbf{r}_{i|A,\overline{a}}$ for each i, there are formulas $\phi_i(\overline{x}, \overline{y}) \in \mathrm{tp}(\overline{a}, \overline{b}/A)$ such that $\mathfrak{C} \not\models \mathrm{d}_{\mathbf{r}_i}\overline{y}\phi[\overline{a}]$ where $\mathrm{d}_{\mathbf{r}_i}\overline{y}\phi \in \mathrm{For}_{\overline{x}}L(A)$ [theorem 2.37]. Then, $\bigwedge_{i=1}^{d} \phi_i(\overline{x}, \overline{y}) \in \mathrm{tp}(\overline{a}, \overline{b}/A)$ and $\mathfrak{C} \models \bigwedge_{i=1}^{d} \neg\, \mathrm{d}_{\mathbf{r}_i}\overline{y}\phi_i[\overline{a}]$. Consider the formula $\psi(\overline{x}, \overline{y}) = \bigwedge_{i=1}^{d} \phi_i(\overline{x}, \overline{y}) \wedge \neg\, \mathrm{d}_{\mathbf{r}_i}\overline{y}\phi_i(\overline{x})$. It is clear that $\psi(\overline{x}, \overline{b}) \in \mathrm{tp}(\overline{a}/A, \overline{b}) \subseteq \mathbf{q}$. Also, for every $\mathbf{q}' \in \mathbf{S}^{\mathfrak{C}}_{\overline{x}}(\mathbf{B})$ such that $\psi(\overline{x}, \overline{b}) \in \mathbf{q}'$, let $\overline{a}'$ be such that ${\mathbf{q}'}_{|A,\overline{b}} = \mathrm{tp}(\overline{a}'/A, \overline{b})$. Then, $\mathfrak{C} \models \psi[\overline{a}', \overline{b}]$, so we have that $\mathrm{tp}(\overline{b}/A, \overline{a}') \neq \mathbf{r}_{i|A,\overline{a}'}$ for each i. Thus, $\overline{b} \not\mathop{\smile\!\!\!\!\!\mid}_A \overline{a}'$, so $\mathbf{q}'$ forks over A. Therefore, $\mathbf{q} \in \langle \psi(\overline{x}, \overline{b}) \rangle \subseteq \mathbf{S}^{\mathfrak{C}}_{\overline{x}}(\mathbf{B}) \setminus \mathrm{NF}(\mathbf{B}/A)$, so $\mathrm{NF}(\mathbf{B}/A)$ is closed. □

Theorem 2.44. *(Open mapping) Let $\mathfrak{C}$ be the monster model of a totally transcendental L-theory and A and B sorted subsets such that $A \subseteq B$. Then, $r_{B/A} : \mathrm{NF}(B/A) \to \mathbf{S}^{\mathfrak{C}}_{\overline{x}}(A)$ defined as $r_{B/A} : p \mapsto p_{|A}$ is open and $r_{\mathbf{C}/A} : \mathrm{NF}(\mathbf{C}/A) \to \mathbf{S}^{\mathfrak{C}}_{\overline{x}}(A)$ is also open.*

Proof. Let $r_{\mathbf{C}/B}$, $r_{\mathbf{C}/A}$ and $r_{B/A}$. It is clear that $r_{\mathbf{C}/A} = r_{B/A} \circ r_{\mathbf{C}/B}$. Let us prove that $r_{\mathbf{C}/A}$ is an open map. Let $U \subseteq \mathrm{NF}(\mathbf{C}/A)$ be an open set and $V =$

$r^{-1}_{\mathbf{C}/A}(r_{\mathbf{C}/A}(U))$. By corollary 2.39, V is the union of all the conjugates of U over A, so V is open. But $\mathbf{S}^{\mathfrak{C}}_{\bar{x}}(A) \setminus r_{\mathbf{C}/A}(U) = r_{\mathbf{C}/A}(\mathrm{NF}(\mathbf{C}/A) \setminus V)$ because $r_{\mathbf{C}/A}$ is an onto function. Since $\mathrm{NF}(\mathbf{C}/A)$ is closed [Theorem 2.43], $r_{B/A}$ is a closed map [proposition 1.19]. Since $r_{\mathbf{C}/A}$ is closed, Im $r_{\mathbf{C}/A}(U)$ is open. Hence, $r_{\mathbf{C}/A}$ is an open map. Finally, let $U \subseteq \mathrm{NF}(B/A)$ be open, then $r_{\mathbf{C}/A}(r^{-1}_{\mathbf{C}/B}(U)) = r_{B/A}(U)$, so $r_{B/A}$ is also open since $r_{\mathbf{C}/A}$ is open and $r_{\mathbf{C}/B}$ is a continuous function. $\square$

2.5 Strongly minimal sets

A pregeometry is a pair (X, cl) such that X is a set, $\mathrm{cl} : \mathcal{P}(X) \to \mathcal{P}(X)$ is a function and every $V \in \mathcal{P}(X)$ satisfies the following properties

1. $V \subseteq \mathrm{cl}(V)$,
2. $\mathrm{cl}(\mathrm{cl}(V)) = \mathrm{cl}(V)$, and
3. **(Finiteness character)** $\mathrm{cl}(V) = \bigcup\{\mathrm{cl}(V_0) : V_0 \subseteq V \text{ finite}\}$,
4. **(Exchange)**$u \in \mathrm{cl}(V \cup \{w\}) \setminus \mathrm{cl}(V) \Rightarrow w \in \mathrm{cl}(V \cup \{u\})$, for every $u, w \in X$.

Note that, by the second property, $\mathrm{cl}(U) \subseteq \mathrm{cl}(V)$ if $U \subseteq V$.

Example. The standard examples are the algebraic closed fields with the algebraic closure and the vector spaces with the linear closure.

Let (X, cl) be a pregeometry and $V \subseteq X$. We say that V is an *independent* set if there is not a proper subset $U \subset V$ such that $\mathrm{cl}(U) = V$; that V is a *generating* set if $\mathrm{cl}(V) = X$, and that V is a *basis* of X if V is an independent and generating set. Note that V is independent if and only if $v \notin \mathrm{cl}(V \setminus \{v\})$ for every $v \in V$.

The following lemmas are fundamental for pregeometries. Their proofs are analogous to the case of vector spaces:

Lemma 2.45. ***(Basis theorem)*** *Let (X, cl) be a pregeometry and $V, W \subseteq X$ subsets such that $V \subseteq W$, W is a generating set and V is an independent set. Then, there is a basis $\mathcal{B} \subseteq X$ such that $V \subseteq \mathcal{B} \subseteq W$.*

Proof. Consider $\Omega = \{U \subseteq W : V \subseteq U \text{ and } U \text{ independent}\} \subseteq \mathcal{P}(W)$. It is clear that $V \in \Omega$, so $\Omega \neq \emptyset$, and that $(\Omega, \subset)$ is a partial order. Let $\Gamma \subseteq \Omega$ be a chain and consider $\bigcup\Gamma$, then $\bigcup\Gamma \in \Omega$. Indeed, for any element $x \in \bigcup\Gamma$, we want to prove that $x \notin \mathrm{cl}(\bigcup\Gamma \setminus \{x\})$. Assume that $x \in \mathrm{cl}(\bigcup\Gamma \setminus \{x\})$. Therefore, there is a finite subset $\Delta \subseteq \bigcup\Gamma$ such that $x \in \mathrm{cl}(\Delta \setminus \{x\})$. Since Δ is finite, there is $U \in \Gamma$ such that $\Delta \cup \{x\} \subseteq U$. If $x \in \mathrm{cl}(\Delta \setminus \{x\})$, then $x \in \mathrm{cl}(U \setminus \{x\})$, a contradiction since U is independent. Then, we apply the Zorn's lemma. Let $\mathcal{B} \in \Omega$ be maximal. Then, $V \subseteq \mathcal{B} \subseteq W$ and $\mathcal{B}$ is independent. Let us prove that $\mathcal{B}$ generates X. Let $w \in V \setminus \mathrm{cl}(\mathcal{B})$, then $\mathcal{B} \cup \{w\} \notin \Omega$ since $\mathcal{B}$ is maximal. Thus, $\mathcal{B} \cup \{w\}$ is not an independent set. Then, there exists $e \in \mathcal{B}$ such that $e \in \mathrm{cl}((\mathcal{B} \setminus \{e\}) \cup \{w\})$. However, $e \notin \mathrm{cl}(\mathcal{B} \setminus \{e\})$ since $\mathcal{B}$ is an independent set. So, by the exchange property, $w \in \mathrm{cl}(\mathcal{B})$. Therefore, $W \subseteq \mathrm{cl}(\mathcal{B})$, so $X = \mathrm{cl}(W) \subseteq \mathrm{cl}(\mathrm{cl}(\mathcal{B})) = \mathrm{cl}(\mathcal{B})$. $\square$

Lemma 2.46. ***(Dimension theorem)*** *Let (X, cl) be a pregeometry and $\mathcal{B}, \mathcal{B}' \subseteq X$ be bases. Then, $\mathrm{card}(\mathcal{B}) = \mathrm{card}(\mathcal{B}')$. We say that $\mathrm{card}(\mathcal{B})$ is the* dimension *of X and we write $\dim X := \mathrm{card}(\mathcal{B})$.*

Proof. Assume that $\mathrm{card}(\mathcal{B}) \leq \mathrm{card}(\mathcal{B}')$. There are two cases, $\mathrm{card}(\mathcal{B}) \geq \aleph_0$ or $\mathrm{card}(\mathcal{B}) \in \mathbb{N}$.

Let $\mathrm{card}(\mathcal{B}) \geq \aleph_0$. Since $\mathcal{B}'$ is a basis, for every $e \in \mathcal{B}$ there is a finite subset $V_e \subseteq \mathcal{B}'$ such that $e \in \mathrm{cl}(V_e)$. Then, $\bigcup_{e\in\mathcal{B}} V_e \subseteq \mathcal{B}'$ is such that $\mathcal{B} \subseteq \mathrm{cl}(\bigcup_{e\in\mathcal{B}} V_e)$, so $\mathrm{cl}(\bigcup_{e\in\mathcal{B}} V_e) = X$. Hence, $\bigcup_{e\in\mathcal{B}} V_e = \mathcal{B}'$. Now, $\mathrm{card}(\mathcal{B}') = \max\{\mathrm{card}(\mathcal{B}), \aleph_0\}$, so $\mathrm{card}(\mathcal{B}') = \mathrm{card}(\mathcal{B})$.

Let $\mathrm{card}(\mathcal{B}) \in \mathbb{N}$. If $\mathcal{B} = \emptyset$, then $\mathcal{B}' = \emptyset$. Assume $\mathcal{B} = \{e_1, \ldots, e_n\}$. Let $E_1, \ldots, E_n \subseteq \mathcal{B}'$ be finite such that $e_i \in \mathrm{cl}(E_i)$ for each $i \in \{1, \ldots, n\}$. Then, $\mathcal{B}' \subseteq \mathrm{cl}(E_1 \cup \cdots \cup E_n)$, so $\mathcal{B}' = E_1 \cup \cdots \cup E_n$. Therefore, $\mathrm{card}(\mathcal{B}') \in \mathbb{N}$. Let $\mathcal{B}' = \{e'_1, \ldots, e'_m\}$. Since $\{e_2, \ldots, e_n\}$ is not a basis, there is $j_1 \in \{1, \ldots, m\}$ such that $e'_{j_1} \notin \mathrm{cl}(\{e_2, \ldots, e_n\})$, so $e_1 \in \mathrm{cl}(\{e'_{j_1}, e_2, \ldots, e_n\})$. Then, $\{e'_{j_1}, e_2, \ldots, e_n\}$ is a basis. Iterate this process and obtain a basis $\{e'_{j_1}, \ldots, e'_{j_n}\} \subseteq \mathcal{B}'$. Hence, $\mathcal{B}' = \{e_{j_1}, \ldots, e'_{j_n}\}$, so $m = n$. $\square$

Let (X, cl) be a pregeometry and $Y \subseteq X$ be a subset. The *restricted pregeometry to Y* is (Y, cl_Y) where $\mathrm{cl}_Y(V) = \mathrm{cl}(V) \cap Y$ for every $V \subseteq Y$. The *locallized pregeometry by Y* is $(X, \mathrm{cl}_{X/Y})$ where $\mathrm{cl}_{X/Y}(V) = \mathrm{cl}(V \cup Y)$ for every $V \subseteq X$, and write $\dim(X/Y) := \dim_{\mathrm{cl}_{X/Y}} X$.

It is a straightforward checkup that (Y, cl_Y) and $(X, \mathrm{cl}_{X/Y})$ are pregeometries.

Lemma 2.47. *Let (X, cl) be a pregeometry and $Y \subseteq X$ be a subset. Then,*

$$\dim(X) = \dim(X/Y) + \dim(Y).$$

Moreover, if $\mathcal{B}_{X/Y}$ is a basis of $(X, \mathrm{cl}_{X/Y})$ and $\mathcal{B}_Y$ is a basis of (Y, cl_Y), then $\mathcal{B}_{X/Y} \cup \mathcal{B}_Y$ is a basis of (X, cl).

Proof. Let $\mathcal{B} = \mathcal{B}_{X/Y} \cup \mathcal{B}_Y$. First of all, I claim that $\mathcal{B}_{X/Y} \cap \mathcal{B}_Y = \emptyset$. Indeed, if $e \in \mathcal{B}_{X/Y} \cap \mathcal{B}_Y$, then $e \in \mathrm{cl}(\mathcal{B}_Y) = \mathrm{cl}(Y)$ and $e \in \mathrm{cl}((\mathcal{B}_{X/Y} \setminus \{e\}) \cup Y)$; a contradiction since $\mathcal{B}_{X/Y}$ is a basis of $(X, \mathrm{cl}_{X/Y})$. Now, we prove that $\mathcal{B}$ is independent. Let $e \in \mathcal{B}$, then either $e \in \mathcal{B}_{X/Y}$ or $e \in \mathcal{B}_Y$. In the first case, if $e \in \mathcal{B}_{X/Y}$, $e \notin \mathrm{cl}((\mathcal{B}_{X/Y} \setminus \{e\}) \cup Y)$, so $e \notin \mathrm{cl}((\mathcal{B}_{X/Y} \setminus \{e\}) \cup \mathcal{B}_Y) = \mathrm{cl}(\mathcal{B} \setminus \{e\})$. In the second case, if $e \in \mathcal{B}_Y \subseteq Y$ and $e \in \mathrm{cl}(\mathcal{B} \setminus \{e\})$, there is a finite subset $W \subseteq \mathcal{B}$ such that $e \in \mathrm{cl}(W)$. We may assume that $e \notin \mathrm{cl}(W')$ for every $W' \subset W$. If $W \subseteq \mathcal{B}_Y$, then $e \in \mathrm{cl}(\mathcal{B}_Y \setminus \{e\}) \cap Y$, which is a contradiction since $\mathcal{B}_Y$ is a basis of (Y, cl_Y). Let $x \in W \cap \mathcal{B}_{X/Y}$, then $e \in \mathrm{cl}(W) \setminus \mathrm{cl}(W \setminus \{x\})$, so $x \in \mathrm{cl}((W \setminus \{x\}) \cup \{e\}) \subseteq \mathrm{cl}((\mathcal{B}_{X/Y} \setminus \{x\}) \cup Y)$, which is a contradiction since $\mathcal{B}_{X/Y}$ is a basis of $(X, \mathrm{cl}_{X/Y})$. Finally, we prove that $\mathcal{B}$ generates X. Indeed, since $Y \subseteq \mathrm{cl}(\mathcal{B}_Y)$, then $\mathcal{B}_{X/Y} \cup Y \subseteq \mathrm{cl}(\mathcal{B}_{X/Y} \cup \mathcal{B}_Y)$, so $X = \mathrm{cl}(\mathcal{B})$. $\square$

A class pregeometry $(\mathbf{X}, \mathrm{cl})$ is a pair such that $\mathbf{X}$ is a class, $\mathrm{cl} : \{V \subseteq \mathbf{X} : V \text{ set}\} \to \{V \subseteq \mathbf{X} : V \text{ set}\}$ is a class function and for every $V \subseteq \mathbf{X}$ set

1. $V \subseteq \mathrm{cl}(V)$,
2. $\mathrm{cl}(\mathrm{cl}(V)) = \mathrm{cl}(V)$ and,
3. **(Finiteness character)** $\mathrm{cl}(V) = \bigcup\{\mathrm{cl}(V_0) \,:\, V_0 \subseteq V \text{ finite}\}$,
4. **(Exchange)** $u \in \mathrm{cl}(V \cup \{w\}) \setminus \mathrm{cl}(V) \Rightarrow w \in \mathrm{cl}(V \cup \{u\})$ for every $u, w \in X$.

Note that $(\mathbf{X}, \mathrm{cl})$ is a class pregeometry if and only if (Y, cl_Y) is a pregeometry for every subset $Y \subseteq \mathbf{X}$. A class pregeometry does not have a basis, but for every subset there is a basis and a dimension.

A *strongly minimal definable set* D of $\mathfrak{M}$ is a definable set such that $\mathrm{MR}(D) = 1$ and $\mathrm{Md}(D) = 1$, i.e., such that every definable subset of it is either finite or cofinite. We have analogous definitions for infinite models (not necessarily $\aleph_0$-saturated) and for definable classes of monster models.

Notation. Let D be a definable set of $\mathfrak{M}$. For $V \subseteq D \subseteq M_{s_1} \times \cdots \times M_{s_n}$, we write $\mathrm{acl}^D(V)$ for the set of elements of D which are tuples from $\mathrm{acl}(V)$. When D is clear from the context, abusing of notation, we write $\mathrm{acl}(V)$ instead of $\mathrm{acl}^D(V)$. We use the same notation for definable classes in monster models.

Theorem 2.48. *Let A be a sorted subset and D be a strongly minimal 0-definable set. Then, (D, acl^D_A) is a pregeometry.*

Proof. Adding A to the language, assume that A is empty. 1. and 3. are clear. For 2., it suffices to prove that $\mathrm{acl}(\mathrm{acl}(V)) \subseteq \mathrm{acl}(V)$. Given $c \in \mathrm{acl}(\mathrm{acl}(V))$, let $\overline{v}$ from V and $\varphi(x, \overline{y}) \in \mathrm{For}\, L$ be such that $\varphi(x, \overline{v})[\mathfrak{M}]$ is a finite set containing c. We may assume that $\varphi(x, \overline{b})[\mathfrak{M}]$ is finite for every $\overline{b}$ — with cardinal at most the number of conjugates of c over $\overline{b}$. On the other hand, let $\phi(\overline{y}) \in \mathrm{For}\, L(B)$ be such that $\overline{b} \in \phi[\mathfrak{M}]$ and $\phi[\mathfrak{M}]$ is finite. Thus, $\psi(x) = \exists \overline{y}(\varphi(x, \overline{y}) \wedge \phi(\overline{y}))$ is an $L(B)$-formula such that $\psi[\mathfrak{M}]$ is finite and $c \in \psi[\mathfrak{M}]$.
Finally, we prove 4., the exchange property. Let $v \in \mathrm{acl}^D(B \cup \{u\}) \setminus \mathrm{acl}^D(B)$. Let B be the sorted of coordinates of elements of B. Then, $\mathrm{MR}(v/B, u) = 0$ and $\mathrm{MR}(v/B) \neq 0$. Note that $\mathrm{MR}(v/B) \leq \mathrm{MR}(D) = 1$. So $v \not\perp_B u$ and $v \not\perp_B u$ by symmetry [Theorem 2.33]. Then, $0 \leq \mathrm{MR}(u/B, v) < \mathrm{MR}(u/B) \leq \mathrm{MR}(D) = 1$ because D is strongly minimal. Hence, $u \in \mathrm{acl}^D(B \cup \{v\})$. $\square$

Corollary 2.49. *Let A be a sorted subset and $\mathbf{D}$ a strongly minimal 0-definable class. Then, $(\mathbf{D}, \mathrm{acl}^{\mathbf{D}}_A)$ is a class pregeometry.*

If $V, U \subseteq \mathbf{D}$ are subsets and A is a sorted subset, we write $\dim(V/U, A)$ for the dimension of V over U in $(\mathbf{D}, \mathrm{acl}^{\mathbf{D}}_A)$. We say that $\dim(V/U, A)$ is the dimension of V over U and A.

Lemma 2.50. *Let $\mathbf{D}$ be a strongly minimal 0-definable class, A a sorted subset and $\overline{v}, \overline{u} \in \mathbf{D}^n$ be tuples such that $\mathrm{MR}(\overline{v}/A) = \mathrm{MR}(\overline{u}/A) = n$. Then, $\mathrm{tp}(\overline{v}/A) = \mathrm{tp}(\overline{u}/A)$.*

Proof. By induction on n. For $n = 1$, by the proposition 2.20,

$$\mathrm{tp}(v_1/A) = \{\varphi \in \mathrm{For}_{\overline{x}} L(A) \,:\, \mathrm{MR}(\mathbf{D} \setminus \varphi[\mathfrak{C}]) < 1\} = \mathrm{tp}(u_1/A).$$

Let the case $n-1$ be proved. Since $\mathrm{tp}(v_1/A) = \mathrm{tp}(u_1/A)$, by the lemma 1.28, there is an automorphism $\mathbf{f}$ fixing A such that $\mathbf{f}(v_1) = u_1$. By induction hypothesis, $\mathrm{tp}(\mathbf{f}(v_2), \ldots, \mathbf{f}(v_n)/A, \mathbf{f}(v_1)) = \mathrm{tp}(u_2, \ldots, u_n/A, u_1)$, so

$$\mathrm{tp}(v_1, \ldots, v_n/A) = \mathrm{tp}(\mathbf{f}(v_1), \ldots, \mathbf{f}(v_n)/A) = \mathrm{tp}(u_1, \ldots, u_n/A).$$

$\square$

Theorem 2.51. ***(Dimension in strongly minimal classes)*** *Let $\mathbf{D}$ be a strongly minimal 0-definable class, $U \subseteq \mathbf{D}$ a subset, A be a sorted subset and $v_1, \ldots, v_n \in \mathbf{D}$ be tuples. Then, $\mathrm{MR}(\overline{v}/U, A) = \dim(\overline{v}/U, A)$.*

Proof. Suppose that $\dim(\overline{v}/U, A) = k$. Rename and let $\{v_1, \ldots, v_k\}$ be a basis of $\{v_1, \ldots, v_n\}$ over U and A. By theorem 2.23, we have that

$$\mathrm{MR}(v_1, \ldots, v_n/U, A) = \mathrm{MR}(v_1, \ldots, v_k/U, A).$$

So, we may assume that $\dim(\overline{v}/U, A) = n$. We prove that $\mathrm{MR}(\overline{v}/A \cup B) = n$ by induction on n. For $n = 1$, since $v_1 \notin \mathrm{acl}_A(U)$ and $v_1 \in \mathbf{D}$, we have that $\mathrm{MR}(v_1/U, A) = 1$. Let the cases $1, \ldots, n-1$ be proved. By induction hypothesis, $\mathrm{MR}(v_2, \ldots, v_n/A, U, v_1) = n-1$, so $\mathrm{MR}(v_1, \ldots, v_n/A, U, v_1) = n-1$ because of theorem 2.23. Since $v_1 \not\perp_{A,U} \overline{v}$, by symmetry [Theorem 2.33], $\mathrm{MR}(\overline{v}/A, U) > n-1$. Suppose that $\mathrm{MR}(\overline{v}/A, U) > n$. Then, there are two disjoint subclasses $\mathbf{E}, \mathbf{F}$ of $\mathbf{D}^n$ of rank n. Let C be a finite sorted subset such that $\mathbf{E}$ and $\mathbf{F}$ are C-definable. Let $\overline{e} \in \mathbf{E}$ and $\overline{f} \in \mathbf{F}$ be generic elements over $A \cup C, U$. Then, $\mathrm{tp}(\overline{e}/A \cup C, U) = \mathrm{tp}(\overline{f}/U, A \cup C)$ by the lemma 2.50. However, $\underline{\mathbf{E}} \in \mathrm{tp}(\overline{e}/U, A \cup C)$, $\underline{\mathbf{F}} \in \mathrm{tp}(\overline{f}/U, A \cup C)$ and $\mathbf{E} \cap \mathbf{F} = \emptyset$, a contradiction. Hence, $\mathrm{MR}(\overline{v}/A, U) = n$. $\square$

Note that, as a consequence of the last theorem, we have that geometric independence is the same that forking independence in strongly minimal definable sets (classes).

Corollary 2.52. ***(Definability of the Morley's rank)*** *Let $B \subseteq \mathbf{C}$ be a sorted subset, $\mathbf{D}$ a strongly minimal 0-definable class and $\phi(\overline{x}, \overline{y}) \in \mathrm{For}\, L(B)$ such that, for every $v \in \mathbf{D}$, $\phi(\overline{x}, v)[\mathfrak{C}] \subseteq \mathbf{D}^n$. Then, for every $k \in \{1, \ldots, n\}$, the class*

$$\{v \in \mathbf{D} \,:\, \mathrm{MR}(\phi(\overline{x}, v)) = k\}$$

is definable.

Proof. Since $\mathrm{MR}(\phi(\overline{x}, v)) \leq n$, it is enough to prove that $\mathbf{X} = \{v \in \mathbf{D} \,:\, \mathrm{MR}(\phi(\overline{x}, v)) \geq k\}$ is definable for each k. We know that $v \in \mathbf{X}$ if and only if there are $u_1, \ldots, u_n \in \mathbf{D}$ tuples such that $\mathfrak{C} \models \phi[\overline{u}, v]$ and $u_{i_1}, \ldots, u_{i_k}$ for some $i_1, \ldots, i_k$ are independent over B, v. So, we have to show that, for any $j_1, \ldots, j_{n-k}$, the class $\mathbf{X}' = \{v \in \mathbf{D} \,:\, \mathrm{MR}(\exists x_{j_1} \ldots x_{j_{n-k}} \phi(\overline{x}, v)) \geq k\}$ is definable. Let $\mathbf{p}^k$ be the global type in $\mathbf{D}^k$ of Morley's rank k, which is unique by lemma 2.50. Then, $\psi(x_{i_1} \ldots x_{i_k}, v) = \exists x_{j_1} \ldots x_{j_{n-k}} \phi(\overline{x}, v)$ has rank greater or equal than k if and only if $\psi \in \mathbf{p}^k$. Since $\mathbf{p}^k$ is definable [corollary 2.29], there is $\mathrm{d}_{\mathbf{p}^k} \overline{x} \psi(\overline{y}) \in \mathrm{For}\, L(\mathbf{C})$ such that $\psi(\overline{x}, v) \in \mathbf{p}^k$ if and only if $\mathfrak{C} \models \mathrm{d}_{\mathbf{p}^k} \overline{x} \psi[v]$. Hence, $\mathbf{X}' = \mathrm{d}_{\mathbf{p}^k} \overline{x} \psi[\mathfrak{C}] \cap \mathbf{D}$. $\square$

An *almost strongly minimal* 0*-definable set* H is a 0-definable set such that there is an strongly minimal 0-definable set D such that $H \subseteq \mathrm{acl}(D)$. Then, we say that H is almost strongly minimal *respect* to D. We have analogous definitions for monster models.

Lemma 2.53. *Let* $\mathbf{D}$ *be a strongly minimal* 0*-definable class,* $\mathbf{H}$ *an almost stronly minimal* 0*-definable class respect to* $\mathbf{D}$ *and* $h \in \mathbf{H}$. *Then, there is a sequence* $u_1, \ldots, u_m, v_1, \ldots, v_n \in \mathbf{D}$ *such that* $\{u_1, \ldots, u_m, v_1, \ldots, v_n\}$ *is an independent set,* h *and* $\overline{v}$ *are interalgebraic over* $\overline{u}$*,* $\{u_1, \ldots, u_m\}$ *is independent over* h *and* $\mathrm{MR}(h/\emptyset) = n$.

Proof. Since $h \in \mathbf{H}$, there is a finite subset $V \subseteq \mathbf{D}$ such that h is algebraic from the coordinates of V. Let $\{v_1, \ldots, v_m\}$ be a basis of (V, acl_h^B) and $\{v_1, \ldots, v_n, u_1, \ldots, u_m\}$ be a basis of V. Let us prove that h and $\overline{v}$ are interalgebraic over $\overline{u}$ and that $n = \mathrm{MR}(h/\emptyset)$.

Since $\{u_1, \ldots, u_m\}$ is a basis of V over h, it is independent over h. Also, we know that $\dim(\overline{u}) = \dim(\overline{u}/h) = m$, so $\mathrm{MR}(\overline{u}) = \mathrm{MR}(\overline{u}/h) = m$ by the theorem 2.51. Thus, $\overline{u} \mathop{\smile\!\!\!\!\!\!\mid\;} h$ and, by symmetry [theorem 2.33], $h \mathop{\smile\!\!\!\!\!\!\mid\;} \overline{u}$. Note that we can apply symmetry since $\mathrm{MR}(h/\emptyset) \leq \mathrm{MR}(\overline{u}, \overline{v}/\emptyset) = n + m$.

Since $\{u_1, \ldots, u_m\}$ is a basis of V over h and $\overline{v}$ is from V, we know that $v_1, \ldots, v_n \in \mathrm{acl}(\overline{u}, h)$. On the other hand, since $\{u_1, \ldots, u_m, v_1, \ldots, v_n\}$ is a basis of V, $V \subseteq \mathrm{acl}(\overline{u}, \overline{v})$. So h is algebraic from the coordinates of $\overline{u}$, $\overline{v}$.

Finally, by the theorem 2.23,

$$\mathrm{MR}(h/\emptyset) = \mathrm{MR}(h/\overline{u}) = \mathrm{MR}(\overline{v}/\overline{u}) = \dim(\overline{v}/\overline{u}) = n.$$

□

Corollary 2.54. *Let* $\mathbf{D}$ *be a strongly minimal* 0*-definable class,* $\mathbf{H}$ *an almost stronly minimal* 0*-definable class respect to* $\mathbf{D}$. *Then,* $\mathrm{MR}(\mathbf{H}) \in \mathbb{N}$.

Proof. Let h be generic in $\mathbf{H}$, by the lemma 2.53, $\mathrm{MR}(\mathbf{H}) = \mathrm{MR}(h/\emptyset) \in \mathbb{N}$. □

Theorem 2.55. ***(Almost strongly minimal)*** *Let* $\mathbf{D}$ *be a strongly minimal* 0*-definable class,* $\mathbf{H}$ *an almost strongly minimal* 0*-definable class respect to* $\mathbf{D}$ *and* $I \subseteq \mathbf{D}$ *an infinite independent set. Then, for every* $h \in \mathbf{H}$*, there is a finite set* $I_0 \subseteq I$ *and finitely many elements* $u_1, \ldots, u_m \in I \setminus I_0$ *and* $v_1, \ldots, v_n \in \mathbf{D}$ *such that* $h \mathop{\smile\!\!\!\!\!\!\mid\;}_{I_0} \overline{u}$*,* $\{u_1, \ldots, u_m\}$ *is independent over* h*,* h *and* $\overline{v}$ *are interalgebraic over* $\overline{u}$ *and* $n = \mathrm{MR}(h/\emptyset)$. *In particular, every element of* $\mathbf{H}$ *is interalgebraic over* I *with a tuple from* $\mathbf{D}$.

Proof. By finiteness [proposition 2.31], let $I_0 \subseteq I$ be a finite subset such that $h \mathop{\smile\!\!\!\!\!\!\mid\;}_{I_0} I$. So $h \mathop{\smile\!\!\!\!\!\!\mid\;}_{I_0} I \setminus I_0$. Let $u \in I \setminus I_0$, then $a \notin \mathrm{acl}((I \setminus \{a\}) \cup \{h\})$. Indeed, $h \mathop{\smile\!\!\!\!\!\!\mid\;}_{I_0} I$ implies $h \mathop{\smile\!\!\!\!\!\!\mid\;}_{I\setminus\{u\}} u$ by monotonicity [proposition 2.31]. By symmetry [Theorem 2.33], $u \mathop{\smile\!\!\!\!\!\!\mid\;}_{I\setminus\{u\}} h$. Now, $\mathrm{MR}(u/I \setminus \{u\}) = \dim(u/I \setminus \{u\})$ by theorem 2.51. Since I is an independent set, $\mathrm{MR}(u/I \setminus \{u\}) = 1$. So $\mathrm{MR}(u/(I \setminus \{u\}) \cup \{h\}) = 1$, i.e., $u \notin \mathrm{acl}((I \setminus \{u\}) \cup \{h\})$. Therefore, $I \setminus I_0$ is an independent set over h. Let $u'_1, \ldots, u'_m, v'_1, \ldots, v'_n \in \mathbf{D}$ be the sequence given by the lemma

2.53. Since v_i' is algebraic over $\overline{u}', h$, let $\varphi_i(x, \overline{u}', h)$, for each i, be such that $\mathfrak{C} \models \varphi_i(x, \overline{u}', h)[v_i]$ and $\text{card}(\varphi_i(x, \overline{u}', h)[\mathfrak{C}]) = k_i \in \mathbb{N}$. For each i, let $\widetilde{\varphi}_i(\overline{u}', h)$ be the sentence which states that $\text{card}(\varphi_i(x, \overline{u}', h)[\mathfrak{C}]) = k_i$. Since h is algebraic from the coordinates of $\overline{u}', \overline{v}'$, let $\psi(z, \overline{u}', \overline{v}')$ be the formula such that $\mathfrak{C} \models \psi(z, \overline{u}', \overline{v}')[h]$ and $\text{card}(\psi(z, \overline{u}', \overline{v}')[\mathfrak{C}]) = N \in \mathbb{N}$. Let $\widetilde{\psi}(h, \overline{u}', \overline{v}')$ be the sentence which states that $\mathfrak{C} \models \psi(z, \overline{u}', \overline{v}')[h]$ and $\text{card}(\psi(z, \overline{u}', \overline{v}')[\mathfrak{C}]) = N$. Let

$$\phi(\overline{y}) = \exists \overline{x} \left(\underline{\mathbf{D}^n}(\overline{x}) \wedge \widetilde{\psi}(h, \overline{y}, \overline{x}) \wedge \bigwedge_{i=1}^{n} (\widetilde{\varphi}_i(\overline{y}, h) \wedge \varphi_i(x_i, \overline{y}, h)) \right).$$

Let $u_1, \ldots, u_m \in I \setminus I_0$, then $\{u_1, \ldots, u_m\}$ is an independent set over h. By the lemma 2.50, $\text{tp}(\overline{u}/h) = \text{tp}(\overline{u}'/h)$. Since $\phi \in \text{tp}(\overline{u}'/h)$, we have that $\phi \in \text{tp}(\overline{u}/h)$. So there are $v_1, \ldots, v_n \in \mathbf{D}$ such that h and $\overline{v}$ are interalgebraic over $\overline{u}$. $\square$

Corollary 2.56. ***(Lascar's equation)*** *Let* $\mathbf{D}$ *be a strongly minimal* 0*-definable class,* $\mathbf{H}$ *an almost strongly minimal* 0*-definable class,* F *a sorted subset and* $g, h \in \mathbf{H}$*. Then,*

$$\text{MR}(g, h/F) = \text{MR}(g/F, h) + \text{MR}(h/F).$$

Proof. By finiteness [proposition 2.31], let F_0 be a finite sorted subset such that $g, h \perp_{F_0} F$, $g \perp_{F_0} F$ and $h \perp_{F_0} F$. Let $V \subseteq \mathbf{D}$ be an infinite independent set over g, h, F_0. By the theorem 2.55, let $\overline{u} \in \mathbf{D}^n$ and $\overline{w} \in \mathbf{D}^m$ be such that $g, \overline{u}$ are interalgebraic over V and $h, \overline{w}$ are interalgebraic over V. Since $\overline{v} \perp g, h, F_0$ for every tuple $\overline{v}$ from V, by monotonicity [proposition 2.31], we know that $\overline{v} \perp_{F_0} g, h$, $\overline{v} \perp_{F_0, h} g$ and $\overline{v} \perp_{F_0} h$ for every tuple $\overline{v}$ from V. Then, by symmetry [Theorem 2.33], we have that $g, h \perp_{F_0} V$, $g \perp_{F_0, h} V$ and $h \perp_{F_0} V$. Since $g, \overline{u}$ are interalgebraic over V and $h, \overline{w}$ are interalgebraic over V, by the theorem 2.23,

$$\begin{aligned}
&\text{MR}(g, h/F_0) = \text{MR}(g, h/F_0 \cup V) = \text{MR}(\overline{u}, \overline{w}/F_0, V), \\
&\text{MR}(g/F_0, h) = \text{MR}(g/F_0, V, h) = \text{MR}(\overline{u}/F_0, V, h), \text{ and} \\
&\text{MR}(h/F_0) = \text{MR}(h/F_0, V) = \text{MR}(\overline{w}/F_0, V).
\end{aligned}$$

Add F_0 and the coordinates of V to the language. Thus, it suffices to prove that

$$\text{MR}(\overline{u}, \overline{w}) = \text{MR}(\overline{u}/h) + \text{MR}(\overline{w}).$$

By the theorem 2.51, we want to prove that

$$\dim(\overline{u}, \overline{w}) = \dim(\overline{u}/h) + \dim(\overline{w}).$$

Since the tuples of $\mathbf{D}$ from $\text{acl}(h)$ are from $\text{acl}(\overline{w})$, that is the same that

$$\dim(\overline{u}, \overline{w}) = \dim(\overline{u}/\overline{w}) + \dim(\overline{w}),$$

which is a particular case of the lemma 2.47. $\square$

A pregeometry or class pregeometry is *modular* if, for every cl-closed sets V and U,

$$\dim(V \cup U) + \dim(V \cap U) = \dim(V) + \dim(U).$$

A pregeometry or class pregeometry is *locally modular* if the locallized pregeometry by w is modular for any $w \notin \mathrm{cl}(\emptyset)$.

Remark. Let (X, cl) be a pregeometry and $V, U \subseteq X$ be cl-closed such that $\dim(U), \dim(V) \in \mathbb{N}$. Then, by the lemma 2.47,

$$\dim(V \cup U) + \dim(V \cap U) = \dim(V) + \dim(U) \Leftrightarrow$$

$$\Leftrightarrow \dim(V \cup U) - \dim(V) = \dim(V) - \dim(U \cap V) \Leftrightarrow$$

$$\Leftrightarrow \dim(U/V) = \dim(U/V \cap U).$$

***Lemma* 2.57.** *Let* (X, cl) *be a pregeometry. Then,* (X, cl) *is modular if and only if* $\dim(V/U) = \dim(V/U \cap V)$ *for any* cl*-closed* $V, U \subseteq X$ *such that* $\dim(V), \dim(U) \in \mathbb{N}$ *and* $\dim(V/U \cap V) = 2$.

Proof. The "only if" part is a particular case. Let us prove the "if" part. Suppose that (X, cl) is not modular. Let

$$n = \min \left\{ \dim(V/V \cap U) \ : \ \begin{array}{l} V, U \subseteq X \text{ cl-closed and} \\ \dim(V) + \dim(U) \neq \dim(V \cap U) + \dim(V \cup U) \end{array} \right\}.$$

Note that $n \in \mathbb{N}$ since the equality is elemental when $\dim(V) \notin \mathbb{N}$ or $\dim(U) \notin \mathbb{N}$. Let $V, U \subseteq X$ be cl-closed such that $\dim(V) + \dim(U) \neq \dim(V \cap U) + \dim(V \cup U)$ and $\dim(V/V \cap U) = n \in \mathbb{N}$. Since $\dim(V), \dim(U) \in \mathbb{N}$,

$$\dim(V \cup U) + \dim(V \cap U) \neq \dim(V) + \dim(U) \Leftrightarrow \dim(V/U) \neq \dim(V/V \cap U).$$

By hypothesis, $n > 2$. Let $\{e_1, \ldots, e_m, e_{m+1}, \ldots, e_n\}$ be basis of V over $V \cap U$ such that $\{e_1, \ldots, e_m\}$ be basis of V over U.
First, we prove that $m = n - 1$. Let $V' = \mathrm{cl}((V \cap U) \cup \{e_1, \ldots, e_{n-1}\})$. Note that $V \cap U = V' \cap U$ since $V \cap U \subseteq V' \subseteq V$. Since n is minimum, then $\dim(V'/U \cap V') = n - 1$ implies that $\dim(V'/U) = n - 1$. Now, $n - 1 = \dim(V'/U) \leq \dim(V/U) < n$, so $m = \dim(V/U) = n - 1$.
Let $U' = \mathrm{cl}(U \cup \{e_1, \ldots, e_{n-2}\})$. I claim that $\{e_{n-1}\}$ is basis of V over U' and $\{e_{n-1}, e_n\}$ is basis of V over $V \cap U'$. It is clear that $V \subseteq \mathrm{cl}_{U'}(e_{n-1})$ and $e_{n-1} \notin \mathrm{cl}(U')$. Also, it is clear that $\mathrm{cl}_{U' \cap V}(e_{n-1}, e_n) = V$. Let us prove that $\{e_{n-1}, e_n\}$ is independent over $U' \cap V$. It suffices to prove that $U' \cap V = \mathrm{cl}((U \cap V) \cup \{e_1, \ldots, e_{n-2}\})$ since $\{e_1, \ldots, e_n\}$ is a basis over $U \cap V$. It is clear that $\mathrm{cl}((U \cap V) \cup \{e_1, \ldots, e_{n-2}\}) \subseteq U' \cap V$. Let us prove that $U' \cap V \subseteq \mathrm{cl}((U \cap V) \cup \{e_1, \ldots, e_{n-2}\})$. Let $w \in V \cap U'$ and $W = \mathrm{cl}(e_1, \ldots, e_{n-2}, w)$. Then, $\dim(W/U \cap W) \leq \dim(W) = n - 1$, so $\dim(W/U) = \dim(W/U \cap W)$. Now, $\dim(W/U) = n - 2$ implies $\dim(W/U \cap W) = n - 2$ and $W \subseteq V$. Therefore, $\dim(W/V \cap U) \leq \dim(W/V \cap W) = n - 2$. Hence, $w \in \mathrm{cl}((U \cap V) \cup \{e_1, \ldots, e_{n-2}\})$.
So $\{e_{n-1}\}$ is a basis of V over U' and $\{e_{n-1}, e_{n-2}\}$ is a basis of V over $V \cap U'$. $\square$

Lemma 2.58. *Let* $\mathbf{D}$ *be a strongly minimal* 0*-definable class such that* $(\mathbf{D}, \mathrm{acl})$ *is modular,* B *an algebraically closed sorted subset and* $\overline{v}$ *be a tuple from* $\mathbf{D}$*. Then,*

$$\overline{v} \underset{\mathrm{acl}(\overline{v})\cap B}{\perp} B.$$

Proof. Let $U \subseteq D$ be the tuples from B. By definition, $\mathrm{acl}(U) = U$. By the theorem 2.51, since $\mathbf{D}$ is modular, we have that

$$\begin{aligned} \mathrm{MR}(\overline{v}/B) =& \dim(v_1, \ldots, v_k/U) = \dim(v_1, \ldots, v_k/U \cap \mathrm{acl}(\overline{v})) = \\ =& \mathrm{MR}(\overline{v}/B \cap \mathrm{acl}(\overline{v})). \end{aligned}$$

So $\overline{v} \perp_{\mathrm{acl}(\overline{v})\cap B} B$. □

Let $\mathfrak{C}$ be the monster model of a totally transcendental L-theory and $\mathbf{D}$ be a 0-definable class of $\mathfrak{C}^{\mathrm{eq}}$. We say that $\mathbf{D}$ is *one-based* if

$$\overline{v} \underset{\mathrm{acl}^{\mathrm{eq}}(\overline{v})\cap \mathrm{acl}^{\mathrm{eq}}(B)}{\perp} B$$

for any tuple $\overline{v}$ from $\mathbf{D}$ and any sorted subset B.

Lemma 2.59. *Let* $\mathfrak{C}$ *be the monster model of a totally transcendental* L*-theory and* $\mathbf{D}$ *be a* 0*-definable class of* $\mathfrak{C}^{\mathrm{eq}}$*. Then,* $\mathbf{D}$ *is one-based if and only if* $\mathrm{cb}(\overline{v}/B) \subseteq \mathrm{acl}^{\mathrm{eq}}(\overline{v})$ *for any sorted subset* B *and any tuple* $\overline{v}$ *from* $\mathbf{D}$ *such that* $\mathrm{tp}(\overline{v}/B)$ *is stationary.*

Proof. ($\Leftarrow$) Let $\overline{v}$ be a tuple from $\mathbf{D}$ with coordinates, B be a sorted subset and $\mathbf{p}$ be the global non-forking extension of $\mathrm{stp}(\overline{v}/B)$. Since $\mathbf{p}$ does not fork over $\mathrm{acl}^{\mathrm{eq}}(B)$, by theorem 2.37, $\mathrm{cb}(\mathbf{p}) \subseteq \mathrm{acl}^{\mathrm{eq}}(B)$. Since $\mathrm{cb}(\mathbf{p}) \subseteq \mathrm{acl}^{\mathrm{eq}}(\overline{v})$, we have that $\mathrm{cb}(\mathbf{p}) \subseteq \mathrm{acl}^{\mathrm{eq}}(\overline{v}) \cap \mathrm{acl}^{\mathrm{eq}}(B)$. By the theorem 2.37, $\overline{v} \perp_{\mathrm{acl}^{\mathrm{eq}}(\overline{v})\cap \mathrm{acl}^{\mathrm{eq}}(B)} \mathrm{acl}^{\mathrm{eq}}(B)$.
($\Rightarrow$) Since $\overline{v} \perp_{\mathrm{acl}^{\mathrm{eq}}(\overline{v})\cap \mathrm{acl}^{\mathrm{eq}}(B)} \mathrm{acl}^{\mathrm{eq}}(B)$, by the theorem 2.37, we have that

$$\mathrm{cb}(\overline{v}/B) \subseteq \mathrm{acl}^{\mathrm{eq}}(\overline{v}) \cap \mathrm{acl}^{\mathrm{eq}}(B) \subseteq \mathrm{acl}^{\mathrm{eq}}(\overline{v}).$$

□

Theorem 2.60. *Let* $\mathfrak{C}$ *be the monster model of a totally transcendental* L*-theory and* $\mathbf{D}$ *be a strongly minimal* 0*-definable class of* $\mathfrak{C}^{\mathrm{eq}}$*. Then, the following are equivalent:*
(1) $(\mathbf{D}, \mathrm{acl}^{\mathbf{D}})$ is a locally modular pregeometry.
(2) $\mathbf{D}$ is one-based.
(3) For any $\mathrm{acl}^{\mathrm{eq}}$-closed sorted subset A and any $v, u \in \mathbf{D}$ such that $\mathrm{MR}(v, u/A) = 1$, $\mathrm{MR}(\mathrm{cb}(v, u/A)) \leq 1$.

Proof. (1)$\Rightarrow$(2) Let $\overline{v}$ be a tuple from $\mathbf{D}$ and B be a finite sorted subset such that $\mathrm{tp}(\overline{v}/B)$ is stationary, and let $\mathbf{p}$ be its non-forking global extension. Let

$w \in \mathbf{D} \setminus \mathrm{acl}(\emptyset)$ be generic over $\overline{v}$ and B, which exists since $\mathrm{MR}(\mathbf{D}) = 1$. Since $(\mathbf{D}, \mathrm{acl}_w)$ is modular, we know that

$$\overline{v} \underset{\mathrm{acl}_w(\overline{v}) \cap \mathrm{acl}_w(B)}{\perp} \mathrm{acl}_w(B).$$

Therefore, $\overline{v} \perp_{\mathrm{acl}^{\mathrm{eq}}(\overline{v},w)\cap \mathrm{acl}^{\mathrm{eq}}(B,w)} B, w$. So, by the theorem 2.37, $\mathrm{cb}(\overline{v}/B, w) \in \mathrm{acl}^{\mathrm{eq}}(\overline{v}, w)$. Since $w \perp B, \overline{v}$, then $\overline{v} \perp_B w$ by monotonicity and symmetry [proposition 2.31 and theorem 2.33]. So $\mathrm{cb}(\overline{v}/B) = \mathrm{cb}(\overline{v}/B, w)$. Since $\mathbf{p}$ does not fork over B, by the theorem 2.37, $\mathrm{cb}(\overline{v}/B) \subseteq \mathrm{acl}^{\mathrm{eq}}(\overline{v}, w) \cap \mathrm{acl}^{\mathrm{eq}}(B)$. Since w is generic over $B, \overline{a}$, we conclude that $\mathrm{acl}^{\mathrm{eq}}(\overline{v}, w) \cap \mathrm{acl}^{\mathrm{eq}}(B) = \mathrm{acl}^{\mathrm{eq}}(\overline{v})$. Indeed, if $d \in \mathrm{acl}^{\mathrm{eq}}(\overline{v}, w) \setminus \mathrm{acl}^{\mathrm{eq}}(\overline{v})$ and $d \in \mathrm{acl}^{\mathrm{eq}}(B)$, then $d \not\perp_{\overline{v}} w$ and, by symmetry [Theorem 2.33], $w \not\perp_{\overline{v}} d$. Since $\mathbf{D}$ is a strongly minimal class, w is from $\mathrm{acl}^{\mathrm{eq}}(\overline{v}, d) \subseteq \mathrm{acl}^{\mathrm{eq}}(\overline{v}, B)$, which is a contradiction since w is generic over $\overline{v}, B$.
(2)$\Rightarrow$(3) Let $d = \mathrm{cb}(v, u/A)$. Since $\mathrm{MR}(v, u/A) = 1$, then $\mathrm{MR}(v, u/d) = 1$. If $v, u \perp d$, then $d \in \mathrm{acl}^{\mathrm{eq}}(\emptyset)$ and $\mathrm{MR}(d) = 0$ [Theorem 2.37]. If $v, u \not\perp d$, by symmetry [Theorem 2.33], $d \not\perp v, u$. Since $\mathbf{D}$ is one-based, by lemma 2.59, $d \in \mathrm{acl}^{\mathrm{eq}}(v, u)$. Thus, $\mathrm{MR}(v, u, d) \leq 2$. By the Lascar's equation [Theorem 2.56], $\mathrm{MR}(v, u, d) = \mathrm{MR}(d) + \mathrm{MR}(v, u/d)$. So $\mathrm{MR}(d) \leq 1$.
(3)$\Rightarrow$(1) Let $w \notin \mathrm{acl}^{\mathbf{D}}(\emptyset)$. By lemma 2.57, it suffices to prove that any $\mathrm{acl}^{\mathbf{D}}_w$-closed sets $V, U \subseteq \mathbf{D}$ such that $\dim(V/U \cap V) = 2$ and $\dim(V/w), \dim(U/w) \in \mathbb{N}$ satisfy $\dim(V/U) = \dim(V/V \cap U)$. Let $V, U \subseteq \mathbf{D}$ be sets with these properties and $\{e_1, e_2\}$ be a basis of V over $V \cap U$. Then, if $\dim(e_1, e_2/U) \neq 2$, $\dim(e_1, e_2/U) < 2$. Firstly, consider that $\dim(e_1, e_2/U) = 0$, then $e_1, e_2 \in \mathrm{acl}^{\mathbf{D}}(U) = U$, so $e_1, e_2 \in V \cap U$ and $\{e_1, e_2\}$ is not a basis over $V \cap U$. Secondly, consider that $\dim(e_1, e_2/U) = 1$ and assume that $\{e_1\}$ is a basis of V over U. By theorem 2.51, $\mathrm{MR}(e_1, e_2/U) = 1$. Assume without lose of generality that U is $\mathrm{acl}^{\mathrm{eq}}_w$-closed [corollary 2.34]. Thus, $\mathrm{tp}(e_1, e_2/B)$ is a stationary type by corollary 2.38. Let $d = \mathrm{cb}(e_1, e_2/B)$, by hypothesis, $\mathrm{MR}(d) = 1$. Since $e_1, e_2 \not\perp d$, by symmetry [Theorem 2.33], $d \not\perp e_1, e_2$. Thus, d is algebraic over the coordinates of e_1, e_2. Since $w \notin \mathrm{acl}^{\mathbf{D}}(e_1, e_2)$, w is not from $\mathrm{acl}(d)$. So w is a generic element of $\mathbf{D}$ over d. On the other hand, by theorem 2.37, $d \in \mathrm{acl}^{\mathrm{eq}}(B)$. Since e_1 is not from $\mathrm{acl}^{\mathrm{eq}}(B)$, e_1 is not from $\mathrm{acl}^{\mathrm{eq}}(d)$. So $e_1 \perp d$ by theorem 2.51, i.e., e_1 is a generic element of $\mathbf{D}$ over d. Since w and e_1 are generic elements of $\mathbf{D}$ over d and $\mathbf{D}$ is strongly minimal, by lemma 2.50, $\mathrm{tp}(w/d) = \mathrm{tp}(e_1/d)$. Let $w' \in \mathbf{D}$ be such that $\mathrm{tp}(w, w'/d) = \mathrm{tp}(e_1, e_2/d)$. Since $\mathrm{MR}(e_1, e_2/d) = 1$, $\mathrm{MR}(w, w'/d) = 1$. So, by theorem 2.51, $\dim(w, w'/d) = 1$. Therefore, $w' \in \mathrm{acl}^{\mathbf{D}}(w, d) \subseteq \mathrm{acl}^{\mathbf{D}}(e_1, e_2, w) \cap \mathrm{acl}^{\mathbf{D}}(U) = V \cap U$. On the other hand, since d is algebraic from the coordinates of e_1, e_2, it is algebraic from the coordinates of w, w'. So, we have that $\mathrm{MR}(e_1, e_2/w, w') = 1$. Hence, by theorem 2.51, $1 = \dim(e_1, e_2/w, w') \geq \dim(V/V \cap U) = 2$, a contradiction. $\square$

We have a criterion to distinguish strongly minimal sets (classes) looking whether its pregeometry is locally modular. Also, we have two examples of strongly minimal sets: vector spaces, which are locally modular, and algebraically closed fields, which are not. Another type of strongly minimal sets (classes) are the *trivial* ones. A strongly minimal set (class) D is trivial if (D, acl^D)

is a trivial (class) pregeometry, and a (class) pregeometry (X, cl) is *trivial* if $\mathrm{cl}(A \cup B) = \mathrm{cl}(A) \cup \mathrm{cl}(B)$ for any pair of subsets A, B. The standard examples of trivial strongly minimal sets are the infinite sets with no structure or the integers with the successor function $(\mathbb{Z}, \bullet + 1)$. Note that strongly minimal groups are not trivial. Indeed, $a \cdot b \in \mathrm{acl}(a, b) \setminus (\mathrm{acl}(a) \cup \mathrm{acl}(b))$ provided $b \notin \mathrm{acl}(a)$.

It is a natural query whether there is an essentially different example. Actually, it is a well-known fact that any non trivial locally modular strongly minimal set (class) arises from vector spaces over a division ring. On the other hand, the case of non locally modular ones, known as Zilber's Conjecture, remained open for a long time: a non locally modular strongly minimal definable set D interprets an algebraically closed field (i.e. an algebraically closed field is definable from D with imaginaries). This conjecture was refuted by Hrushovski who constructed a non locally modular strongly minimal set which does not interpret a group. However, a strong version of the Zilber's conjecture does hold for Zariski geometries.

2.6 Orthogonality

Let D and E be definable sets in $\mathfrak{M}$ with Morley's rank. We say that D and E are orthogonal ($D \perp E$) if $\overline{d} \mathop{\smile\!\!\!\!\!\mid}_A \overline{e}$ for any $\overline{d} \in D$, any $\overline{e} \in E$ and any sorted subset A such that D and E are A-definable. We have analogous definitions for monster models.

Lemma 2.61. *Let $\mathfrak{C}$ be the monster model of a totally transcendental L-theory and $\mathbf{D}, \mathbf{E}$ be two definable classes with Morley's rank. Then, $\mathbf{D} \perp \mathbf{E}$ if and only if $\mathrm{tp}_{\bar{x},\bar{y}}(\overline{d}, \overline{e}/A)$ is the unique complete type extending the partial type $\mathrm{tp}_{\bar{x}}(\overline{d}/A) \cup \mathrm{tp}_{\bar{y}}(\overline{e}/A)$ for every $\overline{d} \in \mathbf{D}$, $\overline{e} \in \mathbf{E}$ and every $\mathrm{acl}^{\mathrm{eq}}$-closed sorted subset A such that $\mathbf{D}$ and $\mathbf{E}$ are A-definable.*

Proof. ($\Leftarrow$) Let $\overline{d} \in \mathbf{D}$, $\overline{e} \in \mathbf{D}$ and A be a sorted subset such that $\mathbf{D}$ and $\mathbf{E}$ are A-definable. Let $A' = \mathrm{acl}^{\mathrm{eq}}(A)$ and $\overline{d}' \in \mathbf{D}$ realize a non-forking extension of $\mathrm{tp}(\overline{d}/A')$ to $A', \overline{e}$. Thus, $\mathrm{tp}_{\bar{x}}(\overline{d}/A') \cup \mathrm{tp}_{\bar{y}}(\overline{e}/A') \subseteq \mathrm{tp}_{\bar{x},\bar{y}}(\overline{d}', \overline{e}/A')$, so, by hypothesis, $\mathrm{tp}(\overline{d}', \overline{e}/A') = \mathrm{tp}(\overline{d}, \overline{e}/A')$ and $\mathrm{tp}(\overline{d}/A', \overline{e}) = \mathrm{tp}(\overline{d}'/A', \overline{e})$. So, $\overline{d} \mathop{\smile\!\!\!\!\!\mid}_{A'} \overline{e}$ since $\overline{d}' \mathop{\smile\!\!\!\!\!\mid}_{A'} \overline{e}$. Now, by corollary 2.34, $\overline{d} \mathop{\smile\!\!\!\!\!\mid}_A A'$, so $\overline{d} \mathop{\smile\!\!\!\!\!\mid}_A \overline{e}$ by transitivity [proposition 2.31].
($\Rightarrow$) Let $\overline{d} \in \mathbf{D}$, $\overline{e} \in \mathbf{D}$ and A be an $\mathrm{acl}^{\mathrm{eq}}$-closed sorted subset such that $\mathbf{D}$ and $\mathbf{E}$ are A-definable. Let d', e' be such that $\mathrm{tp}_{\bar{x}}(\overline{d}/A) \cup \mathrm{tp}_{\bar{y}}(\overline{e}/A) \subseteq \mathrm{tp}_{\bar{x},\bar{y}}(\overline{d}', \overline{e}'/A)$. Since $\mathrm{tp}(\overline{e}/A) = \mathrm{tp}(\overline{e}'/A)$, by lemma 1.28, we may assume that $\overline{e} = \overline{e}'$. It suffices to prove that $\mathrm{tp}(\overline{d}'/A, \overline{e}) = \mathrm{tp}(\overline{d}/A, \overline{e})$. Since $\mathbf{D} \perp \mathbf{E}$, $\mathrm{tp}(\overline{d}/A, \overline{e})$ and $\mathrm{tp}(\overline{d}'/A, \overline{e})$ do not fork over A. There is just one non forking extension of $\mathrm{tp}(\overline{d}/A)$ to $A, \overline{e}$, by theorem 2.38. Hence, $\mathrm{tp}(\overline{d}/A, \overline{e}) = \mathrm{tp}(\overline{d}'/A, \overline{e})$. $\square$

Lemma 2.62. *Let $\mathbf{D}$ be a strongly minimal definable class and $\mathbf{E}$ be a definable class with Morley's rank. Then, $\mathbf{D} \not\perp \mathbf{E}$ if and only if there is a sorted set A*

such that $\mathbf{D} \subseteq \mathrm{acl}(A, \mathbf{E})$. *Moreover, if* $\mathbf{D} \not\perp \mathbf{E}$, *then there is a finite sorted set* A *such that* $\mathbf{D} \subseteq \mathrm{acl}(A, \mathbf{E})$.

Proof. ($\Leftarrow$) If $\mathbf{D} \subseteq \mathrm{acl}(A, \mathbf{E})$ for some sorted set A, let $\overline{d} \in \mathbf{D}$ be generic over A. Let $\overline{e}_1, \ldots, \overline{e}_n \in \mathbf{E}$ be minimal such that $\overline{d} \in \mathrm{acl}(A, \overline{e}_1, \ldots, \overline{e}_n)$. Then, $\overline{d} \not\perp_A \overline{e}_1 \ldots \overline{e}_n$ and $\overline{d} \perp_A \overline{e}_1, \ldots, \overline{e}_{n-1}$. By transitivity [proposition 2.31], we conclude that $\overline{d} \not\perp_{A, \overline{e}_1, \ldots, \overline{e}_{n-1}} \overline{e}_n$. So, $\mathbf{D} \not\perp \mathbf{E}$.
($\Rightarrow$) If $\mathbf{D} \not\perp \mathbf{E}$, there are a sorted set A such that $\mathbf{D}$ and $\mathbf{E}$ is A-definable and $\overline{d} \in \mathbf{D}$ and $\overline{e} \in \mathbf{E}$ such that $\overline{d} \not\perp_A \overline{e}$. By finiteness [proposition 2.31], we may assume that A is finite. Since $\overline{d} \not\perp_A \overline{e}$ and $\mathbf{D}$ is a strongly minimal definable class, $0 \leq \mathrm{MR}(\overline{d}/A, \overline{e}) < \mathrm{MR}(\overline{d}/A) \leq 1$. So $\overline{d}$ is algebraic over $A, \overline{e}$. Now, for any $\overline{d}' \in \mathbf{D}$, either $\mathrm{MR}(\overline{d}'/A) = 0$ or $\mathrm{MR}(\overline{d}'/A) = 1$. Since $\mathbf{D}$ is strongly minimal, by lemma 2.50, if $\mathrm{MR}(\overline{d}'/A) = 1$, $\mathrm{tp}(\overline{d}'/A) = \mathrm{tp}(\overline{d}/A)$. By lemma 1.28, there is an automorphism $\mathbf{f}$ fixing A which maps $\overline{d}$ to $\overline{d}'$. Then, $\mathrm{MR}(\overline{d}'/A, \mathbf{f}(\overline{e})) = 0$. So, for every $\overline{d} \in \mathbf{D}$, there is an element $\overline{e} \in \mathbf{E}$ such that $\overline{d}$ is algebraic over $A, \overline{e}$, i.e., $\mathbf{D} \subseteq \mathrm{acl}(A, \mathbf{E})$. $\square$

Corollary 2.63. *Non-orthogonality is an equivalence relation for strongly minimal classes.*

Corollary 2.64. *Let* $\mathbf{D}$ *be a strongly minimal definable class,* $\mathbf{H}$ *an almost strongly minimal definable class respect to* $\mathbf{D}$ *and* $\mathbf{E}$ *a definable class with Morley's rank. Then,* $\mathbf{H} \not\perp \mathbf{E}$ *if and only if* $\mathbf{D} \not\perp \mathbf{E}$. *Moreover, if* $\mathbf{H} \not\perp \mathbf{E}$, *then there is a finite sorted set* A *such that* $\mathbf{H} \subseteq \mathrm{acl}(A, \mathbf{E})$.

Proof. Assume that $\mathbf{H}$, $\mathbf{D}$ and $\mathbf{E}$ are A-definable.
($\Leftarrow$) Let A be a finite sorted subset such that every element of $\mathbf{D}$ is algebraic over A and a finite tuple of $\mathbf{E}$. Let $h \in \mathbf{H}$, $\overline{d}$ from $\mathbf{D}$ and $\overline{e}$ from $\mathbf{E}$ such that h is not algebraic over A, h is algebraic over $\overline{d}$ and $\overline{d}$ is algebraic over $\overline{e}, A$. Thus, h is algebraic over $\overline{e}, A$ and is not algebraic over A. Hence, $h \not\perp_A \overline{e}$. We may assume that $h \perp_A e_1, \ldots, e_{n-1}$. Indeed, it suffices to consider the minimal $k \leq n$ such that $h \not\perp_A e_1, \ldots, e_k$. Then, $h \not\perp_{A, e_1, \ldots, e_{n-1}} e_n$. Therefore, $\mathbf{H} \not\perp \mathbf{E}$.
($\Rightarrow$) Let A be a finite sorted subset and $h \in \mathbf{H}$ and $e \in \mathbf{E}$ such that $h \not\perp_A e$. By theorem 2.55, there are $\overline{d}$ and $\overline{a}$ from $\mathbf{D}$ such that $h \perp_A \overline{a}$ and h and $\overline{d}$ are interalgebraic over $\overline{a}, A$. Since $h \not\perp_A e$, $h \not\perp_{A, \overline{a}} e$ [proposition 2.31]. Then, by theorem 2.23, $\overline{d} \not\perp_{A, \overline{a}} e$, so $e \not\perp_{A, \overline{a}} \overline{d}$ by symmetry [theorem 2.33]. We may assume that $e \perp_{A, \overline{a}} d_1, \ldots, d_{n-1}$ and $e \not\perp_{A, \overline{a}} d_1, \ldots, d_n$. Thence, $e \not\perp_{A, \overline{a}, d_1, \ldots, d_{n-1}} d_n$, so $\mathbf{E} \not\perp \mathbf{D}$. By lemma 2.62, there is a finite sorted subset A such that every element of $\mathbf{D}$ is algebraic over a finite tuple from $\mathbf{E}$ and A. Thus, every element of $\mathbf{H}$ is algebraic over a finite number of elements of $\mathbf{E}$ and A. $\square$

3 Groups with Morley's rank

A *definable group* is a pair formed by a definable set (or class) and a definable function such that the pair is a group. The parameters of a definable group are the parameters of the set (class) together with the parameters of the operation. The Morley's rank of a definable group $(G, \cdot)$ is $\mathrm{MR}(G)$, and its Morley's degree is $\mathrm{Md}(G)$.

The aim of this chapter is to study the special case of definable groups with Morley's rank. The most significant results studied are the descending chain condition [Theorem 3.3], the characterization of the connected component [Theorem 3.10], Zilber's indecomposability theorem [Theorem 3.16], the properties of one bases groups [Theorem 3.18] and its characterization [Theorem 3.22], and the characterization of orthogonality for groups [theorem 3.28].

Notation. In the rest of the this chapter and except otherwise stated, $(G, \cdot)$ or $(\mathbf{G}, \cdot)$ will denote definable groups with Morley's rank.

3.1 The Descending Chain Condition

Lemma 3.1. *Let $H \leq G$ be a definable subgroup and $a \in G$. Then,* $\mathrm{MR}(H) = \mathrm{MR}(aH) = \mathrm{MR}(Ha)$ *and* $\mathrm{Md}(H) = \mathrm{Md}(aH) = \mathrm{Md}(Ha)$.

Proof. It is a particular case of the corollary 2.25 which implies that definable bijections leave the Morley's rank and degree invariant. □

Lemma 3.2. *Let $H \leq G$ be a definable subgroup. Then, $[G : H]$ is finite if and only if* $\mathrm{MR}(H) = \mathrm{MR}(G)$, *and in that case* $\mathrm{Md}(G) = [G : H] \cdot \mathrm{Md}(H)$. *Moreover, two definable subgroups of G with same Morley's rank and degree and one contained in the other coincide.*

Proof. If $[G : H]$ is infinite, there is a $(a_i)_{i\in\omega}$ sequence of elements in G such that $a_iH \cap a_jH = \emptyset$ for every $i, j \in \omega$ $(i \neq j)$. Then, $\mathrm{MR}(a_iH) = \mathrm{MR}(H)$ for each i implies that $\mathrm{MR}(G) \geq \mathrm{MR}(H) + 1$. On the other hand, if $[G : H] = d$ is finite, there are $a_1, \ldots, a_d$ elements such that $G = a_1H \cup \cdots \cup a_dH$ and $a_iH \cap a_jH = \emptyset$ for each $i \neq j$. Then, $\mathrm{MR}(a_iH) = \mathrm{MR}(H)$ for each i implies that $\mathrm{MR}(G) = \mathrm{MR}(H)$, by the fundamental property 2.11. Also, because these are disjoint, we have that $\mathrm{Md}(G) = \sum_{i=1}^{d} \mathrm{Md}(a_iH) = d \cdot \mathrm{Md}(H) = [G : H] \cdot \mathrm{Md}(H)$.

Finally, let H and H' have the same Morley's rank and the same Morley's degree with $H' \subseteq H$. Since $\mathrm{MR}(H) = \mathrm{MR}(H')$, $[H' : H]$ is finite and $\mathrm{Md}(H') = \mathrm{Md}(H) \cdot [H' : H]$. But $\mathrm{Md}(H') = \mathrm{Md}(H)$, so $[H' : H] = 1$, i.e., $H = H'$. □.

Theorem 3.3. ***(Descending Chain condition)*** *There is no infinite strictly decreasing sequence of definable subgroups G.*

Proof. Suppose there is $(H_i)_{i\in\omega}$, an infinite strictly decreasing sequence of definable subgroups. Since $\mathrm{MR}(G)$ exists, $(\mathrm{MR}(H_i))_{i\in\omega}$ is an infinite decreasing sequence of ordinals. Let $\alpha = \min\{\mathrm{MR}(H_i) : i \in \omega\}$ and $i_0 \in \omega$ be such that $\mathrm{MR}(H_{i_0}) = \alpha$. Then, $\mathrm{MR}(H_i) = \mathrm{MR}(H_{i_0})$ for every $i > i_0$. Consider

$(\mathrm{Md}(H_i))_{i\geq i_0}$, that is a decreasing sequence of non zero natural numbers, so it must have a minimum. Let $i_1 \geq i_0$ be such that $\mathrm{Md}(H_{i_1}) = \min\{\mathrm{Md}(H_i) : i \geq i_0\}$, then we know $H_i \subseteq H_{i_1}$, $\mathrm{MR}(H_i) = \mathrm{MR}(H_{i_1})$ and $\mathrm{Md}(H_i) = \mathrm{Md}(H_{i_1})$ for every $i \geq i_1$. By the last lemma 3.2, $H_i = H_{i_1}$, so $(H_i)_{i\in\omega}$ is not strictly decreasing, a contradiction. □

Corollary 3.4. *The intersection of any family of definable subgroups of G is the intersection of a finite subfamily. In particular, any intersection of definable subgroups of G is definable.*

Example. Let us show a standard application of the descending chain condition. Let A be any subset (not necessarily definable) of elements of G, then the centralizer of A is

$$Z(A) = \{g \in G : \text{ for any } a \in A \;\; g \cdot a = a \cdot g\} = \bigcap_{a\in A} C_G(a).$$

Now, $C_G(a)$ is a definable subgroup since $C_G(a) = G \cap \varphi(\overline{x}, a)[\mathfrak{M}]$ where $\varphi(\overline{x}, \overline{y})$ is $\overline{x} \cdot \overline{y} = \overline{y} \cdot \overline{x}$. Hence, $Z(A)$ is a finite intersection. Therefore, $Z(A) = C_G(a_1) \cap \cdots \cap C_G(a_m)$ for some $\{a_1, \ldots, a_m\} \subseteq A$. Thus, $Z(A) = Z(A_0)$ for some $A_0 \subseteq A$ finite, and this implies that $Z(A)$ is definable.

When A is definable we obtain a sentence of the theory of $\mathfrak{M}$. If $A = \phi[\mathfrak{M}]$, $Z(A) = \psi[\mathfrak{M}] \cap G$, where $\psi(\overline{x}) = \forall \overline{y}(\phi(\overline{y}) \to \varphi(\overline{x}, \overline{y}))$. So, the conclusion is that $\psi[\mathfrak{M}] \cap G = \varphi(\overline{x}, a_1)[\mathfrak{M}] \cap \cdots \cap \varphi(\overline{x}, a_n)[\mathfrak{M}] \cap G$ for some particular $a_1, \ldots, a_n \in A$. Thus,

$$\mathfrak{M} \models \forall \overline{x} \left((\psi(\overline{x}) \wedge \underline{G}(\overline{x})) \leftrightarrow (\varphi(\overline{x}, a_1) \wedge \cdots \wedge \varphi(\overline{x}, a_n) \cap \underline{G}(\overline{x})) \right).$$

Note that this argument is not particular of the centralizer. Indeed, given any family of formulas $\{\varphi_i(\overline{x})\}_{i\in I}$, such that each one defines a subgroup of G, the set of elements satisfying all these formulas is a definable subgroup determined by a finite number of the subgroups. That gives us a useful way to describe many important group-theoretic objects.

3.2 The connected component

The intersection of all the definable subgroups of finite index of G, G°, is the (definable) *connected component* of G. G is *connected* when $G^\circ = G$.

Note that, by the corollary 3.4, G° is a definable subgroup which must be of finite index. Therefore, G° is the smallest definable subgroup of finite index of G.

Note that, by the lemma 3.2, G is connected if $\mathrm{Md}(G) = 1$. We will prove that it is actually an "if and only if" condition.

Proposition 3.5. *If $(G, \cdot)$ is an A-definable, then G° is A-definable too.*

Proof. Add A to the language and assume that $(G, \cdot)$ is 0-definable. Let $\varphi(\overline{x}, \overline{y})$ be an L-formula such that $G^\circ = \varphi(\overline{x}, \overline{a})[\mathfrak{M}]$ where $\overline{a}$ is a tuple of parameters.

Let $k = [G : G^\circ]$, then it is routine to write an L-formula $\psi(\overline{y})$ such that, for any $\overline{b}$, $\mathfrak{M} \models \psi(\overline{b})$ if and only if $\varphi(\overline{x}, \overline{b})[\mathfrak{M}]$ is a subgroup with index k. Then, $\phi(\overline{x}) = \exists \overline{y}(\psi(\overline{y}) \wedge \varphi(\overline{x}, \overline{y}))$ is an L-formula which defines G°. $\square$

Proposition 3.6. *Let $\mathfrak{N}$ be an elementary extension of $\mathfrak{M}$, and $\underline{G}$ and $\underline{G^\circ}$ formulas defining G and G° in $\mathfrak{M}$, respectively. Then, $\underline{G^\circ}[\mathfrak{N}] = \underline{G}[\mathfrak{N}]^\circ$.*

Proof. We know that G° is the unique definable group with index $[G : G^\circ]$. For any formula $\varphi(\overline{x})$, it is expressible by an $L(M)$-formula that $\varphi(\overline{x})$ is a group of index k in G. We conclude by recalling that elementary extensions leave the Morley's rank and the Morley's degree invariant [lemma 2.10]. $\square$

Proposition 3.7. *Let $(G, \cdot)$ be a definable group with or without Morley's rank and $\mathbf{S}_G(M)$ the set of complete types in G. Then, G acts on $\mathbf{S}_G(M)$ by*

$$g \cdot p = \{\varphi(\overline{x}) \,:\, \varphi(g \cdot \overline{x}) \in p\}.$$

Also,
(1) if $a \in G$ realizes $p_{|A}$ and $g \in A$, then $g \cdot p_{|A} = \mathrm{tp}(g \cdot a/A)$;
(2) for any $g \in G$, $\mathrm{MR}(p) = \mathrm{MR}(g \cdot p)$ and, when p has Morley's rank, $\mathrm{Md}(p) = \mathrm{Md}(g \cdot p)$; and
(3) if p has Morley's rank, then Stab_p is a definable subgroup and $\underline{\mathrm{Stab}_p}[\mathfrak{N}] = \mathrm{Stab}_{p'}$ when $\mathfrak{N}$ is an elementary extension of $\mathfrak{M}$ and $p' \in \mathbf{S}^{\mathfrak{M}}(N)$ is the non-forking extension of p.

Proof. A straightforward argument shows that G acts on $\mathbf{S}_G(M)$. (1) It is also clear.
(2) Firstly note that $x \mapsto g \cdot x$ is a definable bijection, so it preserves the Morley's rank and degree [corollary 2.25]. Indeed, for example,

$$\begin{aligned}\mathrm{MR}(p) &= \min\{\mathrm{MR}(\psi) \,:\, \psi \in p\} = \min\{\mathrm{MR}(\psi(g \cdot \overline{x})) \,:\, \psi \in p\} = \\ &= \min\{\mathrm{MR}(\psi) \,:\, \psi \in g \cdot p\} = \mathrm{MR}(g \cdot p).\end{aligned}$$

(3) Assume p has Morley's rank α. Then, given $\phi \in p$ such that $\mathrm{MR}(\phi) = \mathrm{MR}(p) = \alpha$ and $\mathrm{Md}(\phi) = \mathrm{Md}(p)$, we know $p = \{\psi \in \mathrm{For}\ L(M) : \mathrm{MR}(\phi \wedge \neg\psi) < \alpha\}$ and $g \cdot p = \{\psi \in \mathrm{For}\ L(M) \,:\, \mathrm{MR}(\phi(g \cdot \overline{x}) \wedge \neg\psi) < \alpha\}$, by proposition 2.20. Hence, $g \in \mathrm{Stab}_p$ if and only if $\phi(g \cdot \overline{x}) \in p$. Now, since p has Morley's rank, there exists a formula $\mathrm{d}_p\overline{x}\phi(\overline{y} \cdot \overline{x})$ such that $\phi(g \cdot \overline{x}) \in p$ if and only if $\mathfrak{M} \models \mathrm{d}_p\overline{x}\phi(g \cdot \overline{x})$ [corollary 2.29]. Hence, Stab_p is defined by $\mathrm{d}_p\overline{x}\phi(\overline{y} \cdot \overline{x})$. Finally, in the latter case, since p is a global type in $\mathfrak{M}$, by lemma 2.36, there is just one non-forking extension p' to $\mathfrak{N}$, and it is given by $p' = \{\psi \in \mathrm{For}\ L(N) \,:\, \mathrm{MR}(\phi \wedge \neg\psi) < \alpha\}$. Thus, the same formula defines $\mathrm{Stab}_{p'}$ in $\mathfrak{N}$. $\square$

Note that the last proposition can be rewrite for monster models and global types.

Proposition 3.8. *Let $(G, \cdot)$ be a definable group with or without Morley's rank and $p \in \mathbf{S}_G(M)$ a type with Morley's rank. Then, $\mathrm{MR}(\mathrm{Stab}_p) \leq \mathrm{MR}(p)$.*

Proof. Let $A \subseteq M$ be a finite sorted set such that Stab_p is A-definable and p does not forks over A. Let $b \in \mathrm{Stab}_p$ be generic over A. Let $\mathfrak{N}$ be an $|M|^+$-saturated elementary extension of $\mathfrak{M}$ and consider the non-forking extension p' of p to $\mathfrak{N}$ and a an element in N realizing p. Note that $\mathrm{MR}(a/A, b) = \mathrm{MR}(a/A)$, so $\mathrm{MR}(b/A, a) = \mathrm{MR}(b/A)$ by symmetry of Morley's rank independence [Theorem 2.33]. Since $g \mapsto g \cdot a$ is a definable bijection in G, $\mathrm{MR}(b \cdot a/A, a) = \mathrm{MR}(b/A, a)$. Hence,

$$\begin{aligned}\mathrm{MR}(\mathrm{Stab}_p) =& \mathrm{MR}(b/A) = \mathrm{MR}(b/A, a) = \mathrm{MR}(b \cdot a/A, a) \leq \\ \leq& \mathrm{MR}(b \cdot a/A) = \mathrm{MR}((b \cdot p)_{|A}) = \mathrm{MR}(p_{|A}) = \mathrm{MR}(p).\end{aligned}$$

□

Lemma 3.9. *A type $p \in \mathbf{S}_G(M)$ is generic in G if and only if Stab_p has finite index in G. Moreover, $[G : \mathrm{Stab}_p]$ is the number of conjugates of p.*

Proof. If $[G : \mathrm{Stab}_p] \in \mathbb{N}$, then $\mathrm{MR}(\mathrm{Stab}_p) = \mathrm{MR}(G)$ and, by the last proposition 3.8, $\mathrm{MR}(p) = \mathrm{MR}(G)$, so p is generic. On the other hand, when p is generic, $g \cdot p$ is generic too. Hence, $\{g \cdot p : g \in G\}$ is a subset of the finite set of generic types. Now, $g \cdot p = g' \cdot p$ if and only if $g^{-1} \cdot g' \in \mathrm{Stab}_p$, so $\{g \cdot p : g \in G\}$ is in bijection with $\{g\mathrm{Stab}_p : g \in G\}$. Hence, $[G : \mathrm{Stab}_p]$ is finite. □

Remark. Since the set of generic types has cardinal $\mathrm{Md}(G)$, we have proved that $[G : \mathrm{Stab}_p] \leq \mathrm{Md}(G)$ for every p generic in G. Actually, this inequality is an equality, which we will prove it bellow.

Theorem 3.10. ***(Characterization of connected definable groups)*** *G is connected if and only if $\mathrm{Md}(G) = 1$.*

Proof. Of course, $\mathrm{Md}(G) = 1$ if and only if there is just one generic type. We already know that $\mathrm{Md}(G) = 1$ implies that G is connected. We prove the converse, i.e., G connected implies that there is just one generic type. Let p and q be two generic types, we are going to prove that $p = q$. Since Stab_p and Stab_q are subgroups of finite index [lemma 3.10], $\mathrm{Stab}_p = \mathrm{Stab}_q = G$. Let $\mathfrak{N}$ be an $|M|^+$-saturated elementary extension of $\mathfrak{M}$, let a realize p in $\mathfrak{N}$ and q' be the non-forking extension of q to $\mathfrak{N}$. Let $\mathfrak{N}'$ be an $|N|^+$-saturated elementary extension of $\mathfrak{N}$ and b realize q' in $\mathfrak{N}'$. It is clear that $a \in \mathrm{Stab}_{q'}$ since $\mathrm{Stab}_{q'} = \underline{\mathrm{Stab}_q}(\mathfrak{N}') = \underline{G}(\mathfrak{N}')$, where $\underline{\mathrm{Stab}_q}$ and $\underline{G}$ are two formulas which define Stab_q and G in $\mathfrak{M}$, respectively. Then, since a is in N, $\mathrm{tp}(a \cdot b/N) = a \cdot q' = q' = \mathrm{tp}(b/N)$. So, in particular, $\mathrm{tp}(a \cdot b/M) = q$. Now, consider $p^{-1} = \{\varphi(\overline{x}^{-1}) : \varphi \in p\}$ and q^{-1}. By symmetry of Morley's rank independence [Theorem 2.33], the same argument proves that $\mathrm{tp}(b^{-1} \cdot a^{-1}/N, b) = \mathrm{tp}(a^{-1}/N, b)$. Thus, $q = \mathrm{tp}(a \cdot b/M) = \mathrm{tp}(a/M) = p$. □

We have the following corollaries to the last theorem.

Corollary 3.11. *A global type $p \in \mathbf{S}_G(M)$ is generic in G if and only if $\mathrm{Stab}_p = G^\circ$*

Proof. ($\Leftarrow$) By proposition 3.8, $\mathrm{MR}(\mathrm{Stab}_p) \leq \mathrm{MR}(p) \leq \mathrm{MR}(G)$. So, if $\mathrm{Stab}_p = G^\circ$, p is generic.
($\Rightarrow$) Since Stab_p has finite index [lemma 3.9], $G^\circ \leq \mathrm{Stab}(p)$. Let us prove that $\mathrm{Stab}_p \subseteq G^\circ$. Since $[G : G^\circ] = k \in \mathbb{N}$, there are $c_1, \ldots, c_k \in G$ such that

$$\mathfrak{M} \models \forall \overline{x} \left(\underline{G}(\overline{x}) \leftrightarrow \bigvee_{i=1}^{k} \underline{G^\circ}(\overline{x}^{-1} \cdot c_i) \right).$$

Hence, $\underline{G^\circ}(\overline{x}^{-1} \cdot c_i) \in p$ for some $i \in \{1, \ldots, k\}$. Then, $\underline{G^\circ}(a \cdot (\overline{x}^{-1} \cdot c_i)) \in p$ for every $a \in \mathrm{Stab}_p$. So $\underline{G^\circ}(a) \in p$ for every $a \in \mathrm{Stab}_p$, i.e., $\mathrm{Stab}_p \subseteq G^\circ$. $\square$

Corollary 3.12. *The index $[G : G^\circ]$ is equal to $\mathrm{Md}(G)$. Moreover, the orbit of a global generic type in G is the finite set of all the global generic types.*

Proof. Since $\mathrm{Md}(G) = [G : G^\circ]\mathrm{Md}(G^\circ)$, we have that $\mathrm{Md}(G) = [G : G^\circ]$. Therefore, for any generic p, $[G : G^\circ] = [G : \mathrm{Stab}_p]$. So, there are $\mathrm{Md}(G)$ conjugates of p, by lemma 3.9. Now, every conjugate of p is generic in G, and there are $\mathrm{Md}(G)$ generic types in G. Hence, by pigeonhole principle, every generic type is a conjugate of p. $\square$

Proposition 3.13. *Let $\mathfrak{N}$ be an $|M|^+$-saturated elementary extension of $\mathfrak{M}$. Then, every element of G is the product (in $\underline{G}(\mathfrak{N})$) of two elements of $\underline{G}[\mathfrak{N}]$ both generic over M. In particular, if $X \subseteq G$ is a definable set such that $\mathrm{MR}(X) = \mathrm{MR}(G)$ and $\mathrm{Md}(X) = \mathrm{Md}(G)$, $X \cdot X = G$.*

Proof. Let p be a global generic type in G, $a \in G$ and c be an element of $\underline{G}[\mathfrak{N}]$ realizing p, hence generic. Since a is in M, $a \cdot c^{-1}$ and c are interdefinable over M, so $\mathrm{MR}(a \cdot c^{-1}/M) = \mathrm{MR}(c/M)$ by theorem 2.23. Thus, $a \cdot c^{-1}$ is also generic in $\underline{G}[\mathfrak{N}]$ over M and, of course, $a = a \cdot c^{-1} \cdot c$.

Now, if $X \subseteq G$ is a definable set such that $\mathrm{MR}(X) = \mathrm{MR}(G)$ and $\mathrm{Md}(X) = \mathrm{Md}(G)$, then $\mathrm{MR}(G \setminus X) < \mathrm{MR}(G)$ and every generic element in G is in X too. Let $\underline{X}$ be a formula defining X, given $a \in G$, we have proved that

$$\mathfrak{N} \models \exists \overline{x} \exists \overline{y}\ \underline{X}(\overline{x}) \wedge \underline{X}(\overline{y}) \wedge a = \overline{x} \cdot \overline{y}.$$

But $\mathfrak{M} \preceq \mathfrak{N}$, so

$$\mathfrak{M} \models \exists \overline{x} \exists \overline{y}\ \underline{X}(\overline{x}) \wedge \underline{X}(\overline{y}) \wedge a = \overline{x} \cdot \overline{y}.$$

Now, since $a \in G$ was arbitrary, we get $X \cdot X = G$. $\square$

An *infinitely definable group* in $\mathfrak{M}$ is a tuple $(\Sigma, \cdot, -^{-1}, 1)$ such that

1. $\Sigma(\overline{x})$ is a $\overline{s}$-type in $\mathfrak{M}$,
2. $\cdot : (M_{s_1} \times \cdots \times M_{s_n})^2 \to M_{s_1} \times \cdots \times M_{s_n}$ and $\bullet^{-1} : M_{s_1} \times \cdots \times M_{s_n} \to M_{s_1} \times \cdots \times M_{s_n}$ are M-definable functions,
3. $1 \in M_{s_1} \times \cdots \times M_{s_n}$ is such that $\mathfrak{M} \models \Sigma[1]$,
4. $\Sigma(\overline{x}) \cup \Sigma(\overline{y}) \models \Sigma(\overline{x} \cdot \overline{y})$ and $\Sigma(\overline{x}) \models \Sigma(\overline{x}^{-1})$, and

5. $\Sigma(\overline{x}) \models \varphi_1(\overline{x}) \wedge \varphi_2(\overline{x})$ and $\Sigma(\overline{x}) \cup \Sigma(\overline{y}) \cup \Sigma(\overline{z}) \models \psi_3(\overline{x}, \overline{y}, \overline{z})$,

where the formulas $\varphi_1, \varphi_2, \psi_3$ are $1 \cdot \overline{x} = \overline{x} = \overline{x} \cdot 1$, $\overline{x} \cdot \overline{x}^{-1} = \overline{x}^{-1} \cdot \overline{x} = 1$ an $\overline{x} \cdot (\overline{y} \cdot \overline{z}) = (\overline{x} \cdot \overline{y}) \cdot \overline{z}$, respectively.

We say that $(\Sigma, \cdot, -^{-1}, 1)$ is A-definable if Σ is a type with parameters A, the operations $\cdot$ and $-^{-1}$ are A-definable and 1 is A-definable. We write $(\Sigma, \cdot)$.

***Lemma* 3.14.** *Let A be a sorted subset and $(\Sigma, \cdot)$ an infinitely A-definable group with $\mathrm{MR}(\Sigma) \in \mathbb{O}\mathrm{n}$. Then, there is an A-definable set G such that $\Sigma(\overline{x}) \models \underline{G}(\overline{x})$, $\underline{G}(\overline{x})$ isolates $\Sigma(\overline{x})$ and $(G, \cdot)$ is a definable group. In other words, $(\Sigma, \cdot)$ is a definable group.*

Proof. We may assume that Σ is closer under logical consequence. Thus, $\varphi_1, \varphi_2 \in \Sigma$. By Compactness theorem [Theorem 1.11], let $\Delta \subseteq \Sigma$ be finite such that $\Delta(\overline{x}) \cup \Delta(\overline{y}) \cup \Delta(\overline{z}) \models \psi_3(\overline{x}, \overline{y}, \overline{z})$. Thus, $\varphi_3 = \bigwedge_{\psi \in \Delta} \psi \in \Sigma$. Let $\alpha = \mathrm{MR}(\Sigma)$ and $d = \mathrm{Md}(\Sigma)$, and let $\varphi_0 \in \Sigma$ be such that $\mathrm{Md}_\alpha(\varphi_0) = d$. Let $\varphi = \bigwedge_{i=0}^{3} \varphi_i(\overline{x}) \wedge \varphi_i(\overline{x}^{-1})$. It is clear that φ is an A-formula belonging to Σ. Also, it is clear that $\mathrm{Md}_\alpha(\varphi) = d$. Firstly, note that every complete type in $\langle \varphi \rangle_{\mathbf{S}^{\mathfrak{M}}_{\overline{x}}(A)}$ of Morley's rank α extends Σ. Indeed, any complete type p such that $\varphi \in p$ and $\mathrm{MR}(\varphi) = \mathrm{MR}(p)$ is such that $\Sigma \subseteq \{\psi \in \mathrm{For}_{\overline{x}} L(A) \ : \ \mathrm{MR}(\varphi \wedge \neg\psi) < \alpha\} \subseteq p$. Now, let

$$G = \{b \in \varphi[\mathfrak{M}] \ : \ \mathrm{MR}(\varphi(\overline{x}) \wedge \neg\varphi(\overline{x} \cdot b)) < \alpha\}.$$

By theorem 2.28, G is A-definable — note that $G \neq \emptyset$ because $e \in G$. Let us prove that G satisfies the required properties:

i. $\underline{G}$ isolates Σ. Let $b \in G$, we want to prove that $\mathfrak{M} \models \Sigma[b]$. We know that $\mathfrak{M} \models \varphi[b]$. Let $p \in \langle \varphi \rangle_{\mathbf{S}^{\mathfrak{M}}_{\overline{x}}(A,b)}$ be such that $\mathrm{MR}(p) = \alpha$, $\mathfrak{N}$ be an $|M|^+$-saturated elementary extension and a in $\mathfrak{N}$ realize p. Since $\Sigma \subseteq p$, $\mathfrak{N} \models \Sigma[a]$ and $\mathfrak{N} \models \varphi[a]$. Since $b \in G$ and $\{\psi \in \mathrm{For}_{\overline{x}} L(A, b) \ : \ \mathrm{MR}(\varphi \wedge \neg\psi) < \alpha\} \subseteq p$, $\mathfrak{N} \models \varphi[a \cdot b]$. On the other hand, $a \cdot b$ and a are interdefinable over A, b, so $\mathrm{MR}(a \cdot b/A) = \mathrm{MR}(a/A) = \alpha$ [Theorem 2.23]. Then, $\Sigma \subseteq \mathrm{tp}(a \cdot b/A)$. So $\mathfrak{N} \models \Sigma[a]$ and $\mathfrak{N} \models \Sigma[a \cdot b]$. Hence, $\mathfrak{N} \models \Sigma[a^{-1} \cdot (a \cdot b)]$. Finally, since $\mathfrak{N} \models \varphi[a]$, $\mathfrak{N} \models \varphi[a^{-1}]$ and $\mathfrak{N} \models \varphi[b]$, we have that $a^{-1} \cdot (a \cdot b) = b$ and $\mathfrak{N} \models \Sigma[b]$. In particular, $\mathfrak{M} \models \Sigma[b]$.

ii. $\underline{G} \in \Sigma$. Let b in $\mathfrak{M}$ be such that $\mathfrak{M} \models \Sigma[b]$. Let $p_1, \ldots, p_k$ be the types in $\langle \varphi \rangle_{\mathbf{S}^{\mathfrak{M}}_{\overline{x}}(A,b)}$ of Morley's rank α. Let $\mathfrak{N}$ be an $|M|^+$-saturated elementary extension and $a_1, \ldots, a_k$ realize $p_1, \ldots, p_k$ respectively. Then, $\mathfrak{N} \models \Sigma[a_i]$ for each i. Thus, $\mathfrak{N} \models \Sigma[a_i \cdot b]$ for each i. Then, $\varphi(\overline{x} \cdot b) \in p_i$ for each i. So $\mathrm{MR}(\varphi(\overline{x}) \wedge \neg\varphi(\overline{x} \cdot b)) < \alpha$ and $b \in G$. So $\underline{G} \in \Sigma$.

iii. Since $\underline{G}$ is equivalent to Σ, $(G, \cdot)$ is a definable group. $\square$

3.3 Zilber's indecomposability theorem

Let $X \subseteq G$ be a definable subset. We say that X is *indecomposable* if, for every definable subgroup H of G, $X/_H := \{xH \ : \ x \in X\}$ is either infinite or a singleton. We have analogous definitions for monsters models, noted that $\mathbf{X}/_\mathbf{H}$ is a definable class of imaginaries.

An example of indecomposable definable subset is a connected subgroup.

Lemma 3.15. *Let $X \subseteq G$ be a definable subset such that $X^g := gXg^{-1} = X$ for every $g \in G$. Then, X is indecomposable if and only if ${}^{X}/_{K}$ is infinite or a singleton, for every $K \trianglelefteq G$ definable.*

Proof. The "only if" part is trivial. Let us prove the "if" one. Let $K \leq G$ be a definable subgroup such that $\mathrm{card}({}^{X}/_{K}) < \omega$. For every $g \in G$, $\mathrm{card}({}^{X}/_{K^g}) = \mathrm{card}({}^{X^g}/_{K^g}) = \mathrm{card}({}^{X}/_{K})$ because the conjugation by g is a bijection. Consider the normal subgroup $K' = \bigcap_{g\in G} K^g$. By the descending chain condition [corollary 3.4], there is $A \subseteq G$ finite such that $K' = \bigcap_{g\in A} K^g$. So K' is a normal definable subgroup of G. Hence, by hypothesis, ${}^{X}/_{K'}$ is infinite or a singleton. Since $K' = \bigcap_{g\in A} K^g$ is a finite intersection and $\mathrm{card}({}^{X}/_{K^g}) < \omega$ for each $g \in A$, we know that $\mathrm{card}({}^{X}/_{K'}) < \omega$. So ${}^{X}/_{K'}$ is a singleton. Let $x \in X$, so for every $y \in X$, $y^{-1} \cdot x \in K' = \bigcap_{g\in G} K^g$. In particular, $y^{-1} \cdot x \in K$ for every $y \in X$. Hence, ${}^{X}/_{K}$ is a singleton. □

Theorem 3.16. ***(Zilber's indecomposability theorem)*** *Assume G has finite Morley's rank. Let $\{X_i\}_{i\in I}$ be a family of indecomposable definable subsets such that each one contains the identity of G. Then, the group generated by $\bigcup_{i\in I} X_i$ is definable and connected. Moreover, it is generated by the union of finitely many X_i.*

Proof. Assume that $\{X_i\}_{i\in I} = \{X_i^{-1}\}_{i\in I}$. Let $H = \langle \bigcup_{i\in I} X_i \rangle$ be the generated subgroup. For every finite sequence $\bar{t} = (t_1, \dots, t_n) \in {}^{<\omega}I$, consider $X_{\bar{t}} = X_{t_1} \cdots X_{t_n}$. Clearly $X_{\bar{t}} \subseteq H$, for every $\bar{t} \in {}^{<\omega}I$. We have that $\mathrm{MR}(X_{\bar{t}}) \leq \mathrm{MR}(G) < \omega$, so $\{\mathrm{MR}(X_{\bar{t}}) : \bar{t} \in {}^{<\omega}I\}$ is a set of natural numbers less that $\mathrm{MR}(G)$. So, there is a maximum. Let $\bar{t} = (t_1, \dots, t_n) \in {}^{<\omega}I$ be such that $\mathrm{MR}(X_{\bar{t}}) = m = \max\{\mathrm{MR}(X_{\bar{t}}) : \bar{t} \in {}^{<\omega}I\}$. Let p be a global generic type in $X_{\bar{t}}$. I claim that $H = \mathrm{Stab}_p$. We first prove that $H \subseteq \mathrm{Stab}_p$. It suffices to prove that, for every $i \in I$, $X_i \subseteq \mathrm{Stab}_p$. Let $i \in I$. Since $1 \in X_i$, it suffices to prove that ${}^{X_i}/_{\mathrm{Stab}_p}$ is a singleton. Assume that not. Since X_i is indecomposable, ${}^{X_i}/_{\mathrm{Stab}_p}$ is infinite. Let $(a_j)_{j\in\omega} \in {}^{\omega}X_i$ be such that $a_j^{-1} \cdot a_k \notin \mathrm{Stab}_p$ for any $j, k \in \omega$ such that $k \neq j$. Then, $a_k \cdot p \neq a_j \cdot p$ for any $k, j \in \omega$ such that $k \neq j$. On the other hand, we have that, for every $j \in \omega$, $\mathrm{MR}(a_j \cdot p) = m$. Also, by definition of $a_j \cdot p$, $\underline{X_i \cdot X_{\bar{t}}} \in a_j \cdot p$ — indeed, $\underline{X_i \cdot X_{\bar{t}}} \in p$ since $1 \in X_i$. Thus, there are infinitely many types with Morley's rank m in $X_i \cdot X_{\bar{t}} = X_{(i,\bar{t})}$. So $\mathrm{MR}(X_{(i,\bar{t})}) > m$. That contradicts the maximality of m. Now, we prove that $\mathrm{Stab}_p \subseteq H$. Since $H \subseteq \mathrm{Stab}_p$, $X_{\bar{t}} \subseteq \mathrm{Stab}_p$. Therefore, $\mathrm{MR}(X_{\bar{t}}) = \mathrm{MR}(\mathrm{Stab}_p) = \mathrm{MR}(p)$. The latter implies that Stab_p is connected by corollary 3.11, since p is a global generic type in Stab_p. Since $\mathrm{MR}(X) = \mathrm{MR}(\mathrm{Stab}_p)$ and Stab_p is connected, by theorem 3.10, $1 \leq \mathrm{Md}(X) \leq \mathrm{Md}(\mathrm{Stab}_p) = 1$. So $\mathrm{MR}(X) = \mathrm{MR}(\mathrm{Stab}_p)$ and $\mathrm{Md}(X) = \mathrm{Md}(\mathrm{Stab}_p)$. Thence, by proposition 3.13, $\mathrm{Stab}_p = X_{\bar{t}} \cdot X_{\bar{t}} \subseteq H$. The latter also implies that $H = \mathrm{Stab}_p$ is connected and H is generated by $X_{t_1} \cup \cdots \cup X_{t_n}$. □

Remark. For monster models, we can apply the indecomposability theorem to "classes" of indecomposable definable classes.

Corollary 3.17. *Assume G has finite Morley's rank. Then, the commutator $[G,G]$ is definable. Moreover, if G is connected, $[G,G]$ is connected too.*

Proof. $[G,G^\circ]$ is generated by the collection $\{X_a\}_{a\in G}$ where $X_a = \{a\cdot b\cdot a^{-1}\cdot b^{-1} : b\in G^\circ\}$. X_a is definable and $1\in X_a$ for each $a\in G$.

I claim that every X_a is indecomposable. Let $Y_a = \{b\cdot a^{-1}\cdot b^{-1} : b\in G^\circ\}$, since $X_a = a\cdot Y_a$, it suffices to prove that Y_a is indecomposable for each a. Since $b\cdot Y_a\cdot b^{-1} = Y_a$ for every $b\in G^\circ$, by lemma 3.15, it suffices to prove that, for every $K\trianglelefteq G^\circ$ definable, ${}^{Y_a}/_K$ is infinite or a singleton. Now, given $bab^{-1}, cac^{-1}\in Y_a$, we have that $bab^{-1}K = bac^{-1}K$ if and only if $c^{-1}b\in C_{G/_K}(aK)$. So, ${}^{Y_a}/_K$ is infinite or a singleton, since ${}^{G^\circ}/_K$ is connected.

Thus, the indecomposable theorem states that $[G,G^\circ]$ is connected and definable. If G is connected, we have finished. If G is not connected, note that ${}^G/_{[G,G^\circ]}$ has finitely many conjugates, so $\left[{}^G/_{[G,G^\circ]}, {}^G/_{[G,G^\circ]}\right]$ is finite and $[G,G]$ is definable. □

3.4 One-based groups

Notation. In the rest of this chapter and except otherwise stated, $(G,\cdot)$ and $(\mathbf{G},\cdot)$ will be 0-definable groups.

Theorem 3.18. *Let $\mathfrak{C}$ be totally transcendental and $\mathbf{G}$ one-based. Then,*
(1) *for any $n\in\omega$, if $\mathbf{H}\leq\mathbf{G}^n$ is a connected definable subgroup,* $\mathrm{cb}(\mathbf{H})\in\mathrm{acl}^{\mathrm{eq}}(\emptyset)$*;*
(2) *there is a finite definable abelian subgroup of G of finite index; and,*
(3) *for any $\mathbf{p}$ global type in $\mathbf{G}$, there exists $b\in\mathbf{G}$ such that $\underline{\mathrm{Stab}_{\mathbf{p}}\cdot b}\in\mathbf{p}$.*

Proof. (1) Since $\mathbf{H}$ is connected, $\mathrm{Md}(\mathbf{H})=1$, so there is just one global generic type [theorem 3.10]. Let A be a finite sorted subset such that $\mathbf{H}$ is A-definable, $g\in\mathbf{G}^n$ be generic over $\mathrm{acl}^{\mathrm{eq}}(A)$, $\mathbf{p}$ be the global generic type in $\mathbf{H}$ and a be generic in $\mathbf{H}$ over $\mathrm{acl}^{\mathrm{eq}}(A,g)$. Let $q=\mathrm{stp}(g\cdot a/A,g)$, $u=\mathrm{cb}(\mathbf{H})$ and $v=\mathrm{cb}(q)$. Note the following:
i. $v\in\mathrm{acl}^{\mathrm{eq}}(g\cdot a)$ since $\mathbf{G}$ is one-based [Lemma 2.59].
ii. $u\in\mathrm{acl}^{\mathrm{eq}}(A)$ since $\mathbf{H}$ is A-definable [Theorem 1.29].
iii. $\mathrm{stp}(g\cdot a/A)$ is generic in $\mathbf{G}^n$. Indeed, we know that $a\mathop{\smile\!\!\!\!\!|}_{\mathrm{acl}^{\mathrm{eq}}(A)} g$ and, by symmetry [Theorem 2.33], $g\mathop{\smile\!\!\!\!\!|}_{\mathrm{acl}^{\mathrm{eq}}(A)} a$. Therefore, since g is generic over $\mathrm{acl}^{\mathrm{eq}}(A)$ and $g\cdot a$ and g are interdefinable over $\mathrm{acl}^{\mathrm{eq}}(A),a$, by theorem 2.23, we have

$$\begin{aligned}\mathrm{MR}(g\cdot a/\mathrm{acl}^{\mathrm{eq}}(A),a) =&\mathrm{MR}(g/\mathrm{acl}^{\mathrm{eq}}(A),a) = \mathrm{MR}(g/\mathrm{acl}^{\mathrm{eq}}(A)) =\\ =&\mathrm{MR}(\mathbf{G}^n)\geq\mathrm{MR}(g\cdot a/\mathrm{acl}^{\mathrm{eq}}(A))\geq\mathrm{MR}(g\cdot a/\mathrm{acl}^{\mathrm{eq}}(A),a).\end{aligned}$$

iv. $g\cdot a\mathop{\smile\!\!\!\!\!|}_{\emptyset} u$. Indeed, by **ii.** and **iii.**,

$$\begin{aligned}\mathrm{MR}(\mathbf{G}^n)\geq&\mathrm{MR}(g\cdot a)\geq\mathrm{MR}(g\cdot a/u)\geq\\ \geq&\mathrm{MR}(g\cdot a/\mathrm{acl}^{\mathrm{eq}}(A)) = \mathrm{MR}(\mathbf{G}^n).\end{aligned}$$

v. $u \in \mathrm{dcl}^{\mathrm{eq}}(v)$. Indeed, we use the theorem 1.29. Let $\mathbf{f}$ be an automorphism fixing v and let $\mathbf{H}' = \mathbf{f}(\mathbf{H})$ and $g' = \mathbf{f}(g)$. Since $\underline{g\mathbf{H}}$ and $\underline{g'\mathbf{H}'}$ both belong to q and $g \cdot a$ and a are interdefinable over $\mathrm{acl}^{\mathrm{eq}}(A, g)$, by theorem 2.23,

$$\begin{aligned}\mathrm{MR}(q) =& \mathrm{MR}(g \cdot a/\mathrm{acl}^{\mathrm{eq}}(A, g)) = \mathrm{MR}(a/\mathrm{acl}^{\mathrm{eq}}(A, g)) = \\ =& \mathrm{MR}(\mathbf{H}) \geq \mathrm{MR}(g\mathbf{H} \cap g'\mathbf{H}') \geq \mathrm{MR}(q).\end{aligned}$$

Since $g\mathbf{H} \cap g'\mathbf{H}' = g''(\mathbf{H} \cap \mathbf{H}')$ for any $g'' \in g\mathbf{H} \cap g'\mathbf{H}'$, we have that $\mathrm{MR}(\mathbf{H} \cap \mathbf{H}') = \mathrm{MR}(\mathbf{H})$. But $\mathbf{H}$ is connected, $\mathbf{H} = \mathbf{H} \cap \mathbf{H}'$, so $\mathbf{H} \subseteq \mathbf{H}'$. Also, $\mathbf{H}'$ is connected and $\mathrm{MR}(\mathbf{H}) = \mathrm{MR}(\mathbf{H}')$, so $\mathbf{H} = \mathbf{H}'$.

Finally, **v.** and **i.** together imply that $u \in \mathrm{acl}^{\mathrm{eq}}(g \cdot a)$, and by **iv.** we get that $u \in \mathrm{acl}^{\mathrm{eq}}(\emptyset)$.

(2) It suffices to prove that $\mathbf{G}^\circ$ is abelian, so we may assume that $\mathbf{G}$ is connected — observe that $\mathbf{G}^\circ$ is also one-based. Consider $\mathbf{G}^2$ and, for any $g \in \mathbf{G}$, $\mathbf{H}_g = \{(h, g^{-1} \cdot h \cdot g) : h \in \mathbf{G}\} \leq \mathbf{G}^2$. Since $\mathbf{H}_g$ and $\mathbf{G}$ are definably isomorphic, $\mathbf{H}_g$ is connected [theorem 2.25 and theorem 3.10]. Consider the definable equivalence relation $g \sim g' \Leftrightarrow \mathbf{H}_g = \mathbf{H}_{g'}$. Of course, $g \sim g'$ if and only if $g^{-1} \cdot h \cdot g = g'^{-1} \cdot h \cdot g'$ for every $h \in \mathbf{G}$. Therefore, $g \sim g'$ if and only if $g^{-1} \cdot g' \in Z(\mathbf{G})$. Now, by (1), $\mathbf{H}_g$ is definable by a finite tuple from $\mathrm{acl}^{\mathrm{eq}}(\emptyset)$ for every $g \in \mathbf{G}$. Thus, there are at most $\mathrm{card}(L)$ different $\mathbf{H}_g$, i.e., $[\mathbf{G} : Z(\mathbf{G})] \leq \mathrm{card}(L)$. However, in $\mathfrak{C}$, the latter is possible if and only if $[\mathbf{G} : Z(\mathbf{G})] < \omega$. Hence, $\mathbf{G} = Z(\mathbf{G})$ since $\mathbf{G}$ is connected.

(3) Let $A_0 = \mathrm{cb}(\mathbf{p})$, g be generic in $\mathbf{G}$ over A_0 and $\mathbf{q} = g \cdot \mathbf{p}$. Let $u = \mathrm{cb}(g\mathrm{Stab}_{\mathbf{p}})$ and $v = \mathrm{cb}(\mathbf{q})$. Firstly, note that u and v are interdefinable over A_0 by theorem 1.29. Indeed, every automorphism $\mathbf{f}$ fixing A_0 leaves $\mathbf{p}$ invariant, so leaves $\mathrm{Stab}_{\mathbf{p}}$ invariant too. Then, $\mathbf{f}$ fixes u if and only if $\mathbf{f}(g)\mathrm{Stab}_{\mathbf{p}} = g\mathrm{Stab}_{\mathbf{p}}$. So $\mathbf{f}$ fixes u if and only if $\mathbf{q} = g \cdot \mathbf{p} = \mathbf{f}(g) \cdot \mathbf{p} = \mathbf{f}(\mathbf{q})$. Therefore, $\mathbf{f}$ fixes u if and only if $\mathbf{f}$ fixes v. Let $\mathfrak{M} \prec \mathfrak{C}$ be an $\aleph_0$-saturated structure where A_0 and g are in and a be such that $\mathrm{tp}(a/M) = \mathbf{p}_{|M}$. Then, since $g \in M$, $\mathbf{q}_{|M} = \mathrm{tp}(g \cdot a/M)$. Since $\mathrm{tp}(g \cdot a/M)$ is stationary and $\mathbf{G}$ is one-based, $v = \mathrm{cb}(\mathbf{q}) = \mathrm{cb}(g \cdot a/M) \in \mathrm{acl}^{\mathrm{eq}}(g \cdot a)$ by lemma 2.59. So $u \in \mathrm{acl}^{\mathrm{eq}}(A_0, g \cdot a)$. Now, $a \mathop{\smile\!\!\!\!\!\!\;|}_{A_0} g$ and g is generic in $\mathbf{G}$ over A_0. So $g \cdot a \mathop{\smile\!\!\!\!\!\!\;|}_{A_0} a$ and $g \cdot a$ is generic in $\mathbf{G}$ over A_0. Indeed, by theorem 2.23,

$$\begin{aligned}\mathrm{MR}(\mathbf{G}) \geq& \mathrm{MR}(g \cdot a/A_0) \geq \mathrm{MR}(g \cdot a/A_0, a) = \\ =& \mathrm{MR}(g/A_0, a) = \mathrm{MR}(g/A_0) = \mathrm{MR}(\mathbf{G}).\end{aligned}$$

Since $u \in \mathrm{acl}^{\mathrm{eq}}(A_0, g \cdot a)$, we conclude by symmetry [theorem 2.33 and corollary 2.34] that $a \mathop{\smile\!\!\!\!\!\!\;|}_{A_0} g \cdot a, u$. Let $b \in M$ realize $\mathbf{p}_{|u, A_0}$. So, $b \mathop{\smile\!\!\!\!\!\!\;|}_{A_0} u$. We know that $v = \mathrm{cb}(\mathbf{q}) \in \mathrm{dcl}^{\mathrm{eq}}(A_0, u)$, so $\mathbf{q}$ does not fork over A_0, u [theorem 2.37]. Also, $\mathrm{tp}(g \cdot a/M) = \mathbf{q}_{|M}$, so $g \cdot a \mathop{\smile\!\!\!\!\!\!\;|}_{A_0, u} b$ by monotonicity [proposition 2.31]. By symmetry [theorem 2.33], $b \mathop{\smile\!\!\!\!\!\!\;|}_{A_0, u} g \cdot a$. By transitivity [proposition 2.31], $b \mathop{\smile\!\!\!\!\!\!\;|}_{A_0} g \cdot a, u$. Hence, $a \mathop{\smile\!\!\!\!\!\!\;|}_{A_0} g \cdot a, u$, $b \mathop{\smile\!\!\!\!\!\!\;|}_{A_0} g \cdot a, u$ and $\mathrm{tp}(b/A_0) = \mathrm{tp}(a/A_0) =$

$\mathbf{p}_{|A_0}$. But $A_0 = \mathrm{cb}(\mathbf{p})$, so $\mathbf{p}_{|A_0}$ is stationary [corollary 2.40]. Therefore, $\mathrm{tp}(b/A_0, g \cdot a, u) = \mathrm{tp}(a/A_0, g \cdot a, u)$. We know that $g \cdot a \in g\mathrm{Stab}_{\mathbf{p}} a$. So $\underline{g\mathrm{Stab}_{\mathbf{p}}}((g \cdot a) \cdot \overline{x}^{-1}) \in \mathrm{tp}(a/A_0, g \cdot a, u) = \mathrm{tp}(b/A_0, g \cdot a, u)$. So $g \cdot a \in g\mathrm{Stab}_{\mathbf{p}} b$. In particular, $a \in \mathrm{Stab}_{\mathbf{p}} b$. Hence, we conclude that $\underline{\mathrm{Stab}_{\mathbf{p}}} b \in \mathrm{tp}(a/M) = \mathbf{p}_{|M} \subseteq \mathbf{p}$. □

Our next aim is to prove the theorem 3.22, which is a characterization of one-based 0-definable groups. To do that, we need the following lemmas.

Lemma 3.19. *Let L be an S-language, T a complete L-theory, $\phi \in \mathrm{For}_{\overline{x}} L$ consistent with T and $\mathcal{F} \subseteq \mathrm{For}_{\overline{x}} L$ a non-empty set closed under $\wedge, \vee$ and $\neg$ such that $\phi \in \mathcal{F}$ and there is $\varphi \in \mathcal{F}$ with $\varphi \in p$ and $\varphi \notin q$ for every pair of different types $p, q \in \langle \phi \rangle$. Then, every formula $\varphi \in \mathrm{For}_{\overline{x}} L$ such that $T \models \forall \overline{x}(\varphi \to \phi)$ is equivalent to a formula of $\mathcal{F}$ modulo T.*

Proof. Of course, $\langle \varphi \rangle \subseteq \bigcap \{ \langle \psi \rangle \ : \ \psi \in \mathcal{F} \text{ and } T \models \forall \overline{x}(\varphi \to \psi)\}$. Let us prove by contradiction that

$$\langle \varphi \rangle = \bigcap_{\substack{\psi \in \mathcal{F} \\ T \models \forall \overline{x}(\varphi \to \psi)}} \langle \psi \rangle.$$

Let $p \in \mathbf{S}_{\overline{x}}(T)$ be such that $\varphi \notin p$ and $\psi \in p$ for every $\psi \in \mathcal{F}$ such that $T \models \forall \overline{x}(\varphi \to \psi)$. In particular, $\phi \in p$. For every $q \in \langle \varphi \rangle$, since $\phi \in q$, there is $\psi_q \in \mathcal{F}$ such that $\psi_q \in p$ and $\psi_q \notin q$. Let $\Sigma = \{\psi_q\}_{q \in \langle \varphi \rangle} \subseteq \mathcal{F}$. Thus, $\Sigma \subseteq p$ and $\langle \Sigma \rangle \cap \langle \varphi \rangle = \emptyset$. By the Compactness theorem [Theorem 1.11], there is $\Delta \subseteq \Sigma$ finite such that $\langle \Delta \rangle \cap \langle \varphi \rangle = \emptyset$. Therefore, $\psi_0 = \bigwedge_{\psi \in \Delta} \psi$ is such that $\psi_0 \in \mathcal{F}$, $\psi_0 \in p$ and $T \models \forall \overline{x}(\varphi \to \neg \psi_0)$. So, $\neg \psi_0 \in p$ and $\psi_0 \in p$, which is a contradiction. Hence,

$$\langle \varphi \rangle = \bigcap_{\substack{\psi \in \mathcal{F} \\ T \models \forall \overline{x}(\varphi \to \psi)}} \langle \psi \rangle.$$

By compactness of $\mathbf{S}_{\overline{x}}(T)$ [proposition 1.18], there is a finite subset $\Delta \subseteq \mathcal{F}$ such that $\langle \varphi \rangle = \langle \Delta \rangle = \langle \bigwedge_{\psi \in \Delta} \psi \rangle$, and $\bigwedge_{\psi \in \Delta} \psi \in \mathcal{F}$. □

Lemma 3.20. *Let $\mathfrak{C}$ be totally transcendental. Then, for every n, every definable subclass of $\mathbf{G}^n$ is a boolean combination of cosets of $\mathrm{acl}(\emptyset)$-definable subgroups of $\mathbf{G}^n$ if and only if, for every n, every definable subclass of $\mathbf{G}^n$ is a finite boolean combination of cosets of definable subgroups of $\mathbf{G}^n$.*

Proof. The "only if" part is clear, let us prove the "if" part. It suffices to prove that every definable subgroup has a $\mathrm{acl}(\emptyset)$-definable subgroup of finite index. Let $\mathbf{H}_{\overline{a}} \subseteq \mathbf{G}^n$ be an $\overline{a}$-definable subclass. Let $\varphi(\overline{x}, \overline{y}) \in \mathrm{For}_{\overline{x}, \overline{y}} L$ be such that $\varphi(\overline{x}, \overline{a})[\mathfrak{C}] = \mathbf{H}_{\overline{a}}$. Let

$$\mathbf{H} = \{(\overline{c}, \overline{d}) \ : \models \varphi(\overline{c}, \overline{d}) \text{ and } \varphi(\overline{x}, \overline{d})[\mathfrak{C}] \leq \mathbf{G}\}$$

and write $\mathbf{H}_{\overline{d}} = \{\overline{c} \ : \ (\overline{c}, \overline{d}) \in \mathbf{H}\}$. It is clear that $\mathbf{H} \subseteq \mathbf{G}^{n+m}$ is a 0-definable class. By hypothesis, $\mathbf{H}$ is a boolean combination of cosets of definable subgroups. Therefore,

$$\mathbf{H} = \bigcup_{i=0}^{N} \left(\bigcap_{k=0}^{M_i'} \overline{c}_{ik}\mathbf{K}_{ik} \setminus \bigcup_{j=0}^{M_i} \overline{d}_{ij}\mathbf{F}_{ij} \right)$$

where $\mathbf{F}_{ij}, \mathbf{K}_{ik} \leq \mathbf{G}^{n+m}$ for each i, j, k. The intersection of cosets is empty or a coset, so we may assume that $M_i' = 0$. Thus,

$$\mathbf{H} = \bigcup_{i=0}^{N} \left(\overline{c}_i\mathbf{K}_i \setminus \bigcup_{j=0}^{M_i} \overline{d}_{ij}\mathbf{F}_{ij} \right)$$

Let $\mathbf{D}_i := c_i\mathbf{K}_i$ and $\mathbf{E}_{ij} := d_{ij}\mathbf{F}_{ij}$ for each i, j. Let $\overline{b}$ be such that $\mathbf{D}_i$, $\mathbf{K}_i$, $\mathbf{E}_{ij}$ and $\mathbf{F}_{ij}$ are $\overline{b}$-definable for each i, j. Since $\mathbf{H}$ is 0-definable, we may assume that that $\overline{b} \perp_{\emptyset} \overline{a}$. Let $\overline{h} \in \mathbf{H}^{\circ}_{\overline{a}}$ be a generic element over $\overline{a}, \overline{b}$. Since $(\overline{h}, \overline{a}) \in \mathbf{H}$, there is an i such that $(\overline{h}, \overline{a}) \in \mathbf{D}_i \setminus \bigcup_{j=0}^{M_i} \mathbf{E}_{ij}$. We may assume that $(\overline{h}, \overline{a}) \in \mathbf{D}_0 \setminus \bigcup_{j=0}^{M_i} \mathbf{E}_{0,j}$. Write $\overline{c} := \overline{c}_0$, $\overline{d}_j := \overline{d}_{0,j}$, $\mathbf{F}_j := \mathbf{F}_{0,j}$, $\mathbf{K} := \mathbf{K}_0$, $\mathbf{D} := \mathbf{D}_0$ and $\mathbf{E}_j := \mathbf{E}_{0,j}$ for each j. Let $\mathbf{K}' = \{\overline{u} : (\overline{u}, 1) \in \mathbf{K}\}$ and $\mathbf{F}_j' = \{\overline{u} : (\overline{u}, 1) \in \mathbf{F}_j\}$ for each j. I claim that $\mathbf{H}^{\circ}_{\overline{a}} = \mathbf{K}'^{\circ}$.

Firstly, let us prove that $\mathbf{H}^{\circ}_{\overline{a}} \subseteq \mathbf{K}'^{\circ}$. Let $\overline{e}$ be a generic element in $\mathbf{H}^{\circ}_{\overline{a}}$ over $\overline{a}, \overline{b}, \overline{h}$. Note that $\overline{h} \cdot \overline{e}$ is a generic element in $\mathbf{H}^{\circ}_{\overline{a}}$ over $\overline{a}, \overline{b}, \overline{h}$ [Theorem 2.23]. Since $\mathbf{H}^{\circ}_{\overline{a}}$ is connected, $\mathrm{tp}(\overline{h} \cdot \overline{e}/\overline{a}, \overline{b}) = \mathrm{tp}(\overline{h}/\overline{a}, \overline{b})$ because both are generic. Since $(\overline{h}, \overline{a}) \in \mathbf{D}$ and $\mathbf{D}$ is $\overline{b}$-definable, then $(\overline{h} \cdot \overline{e}, \overline{a}) \in \mathbf{D}$. So $(\overline{h}, \overline{a}), (\overline{h} \cdot \overline{e}, \overline{a}) \in \overline{c}\mathbf{F}$. Hence, $(\overline{e}, 1) \in \mathbf{K}$, i.e., $\overline{e} \in \mathbf{K}'$. Therefore, we have proved that every generic element of $\mathbf{H}^{\circ}_{\overline{a}}$ is also in $\mathbf{K}'$. By proposition 3.13, we conclude that $\mathbf{H}^{\circ}_{\overline{a}} \subseteq \mathbf{K}'$, so $\mathbf{H}^{\circ}_{\overline{a}} \subseteq \mathbf{K}'^{\circ}$.

Now, we prove that $\mathbf{K}'^{\circ} \subseteq \mathbf{H}^{\circ}_{\overline{a}}$. Let $Q_1 = \{j : (1, \overline{a}) \in \mathbf{E}_j\}$ and $Q_2 = \{j : [\mathbf{K}' : \mathbf{F}_j'] < \omega\}$. Then, $Q_1 \cap Q_2 = \emptyset$. Indeed, if $j \in Q_1 \cap Q_2$, then $\overline{h} \in \mathbf{H}^{\circ}_{\overline{a}} \subseteq \mathbf{K}'^{\circ} \subseteq \mathbf{F}_j'$. So $(\overline{h}, 1) \in \mathbf{F}_j$ and $(1, \overline{a}) \in \mathbf{E}_j$. Therefore, $(\overline{h}, \overline{a}) \in \mathbf{E}_j$, a contradiction since $(\overline{h}, \overline{a}) \notin \mathbf{E}_j$. So, $Q_1 \cap Q_2 = \emptyset$. Let q be the generic type in $\mathbf{K}'^{\circ}$ over $\overline{a}, \overline{b}, \overline{h}$. For each j, since $\mathbf{E}_j$ is $\overline{b}$-definable, either $\neg\underline{\mathbf{E}_j}(\overline{x}, \overline{a}) \in q$ or $\underline{\mathbf{E}_j}(\overline{x}, \overline{a}) \in q$ for some j.

Let us prove that $\underline{\mathbf{E}_j}(\overline{x}, \overline{a}) \notin q$ for each j. Indeed, if $\underline{\mathbf{E}_j}(\overline{x}, \overline{a}) \in q$, for every generic element $\overline{e} \in \mathbf{K}'^{\circ}$ over $\overline{a}, \overline{b}, \overline{h}$, $(\overline{e}, \overline{a}) \in \mathbf{E}_j$. Let $\overline{e}, \overline{e'}$ be independent generic elements in $\mathbf{K}'^{\circ}$ over $\overline{a}, \overline{b}, \overline{h}$, then $\overline{e'} \cdot \overline{e}$ is a generic element over $\overline{a}, \overline{b}, \overline{c}$ too [Theorem 2.23]. Thus, $(\overline{e}, \overline{a}), (\overline{e}', \overline{a}), (\overline{e}' \cdot \overline{e}, \overline{a}) \in \mathbf{E}_j$. So $\overline{e} \in \mathbf{F}_j'$. Since $\overline{e}$ is arbitrary, every generic element of $\mathbf{K}'^{\circ}$ belongs to $\mathbf{F}_j'$. By proposition 3.13, we conclude that $\mathbf{K}'^{\circ} \subseteq \mathbf{F}_j'$. So $[\mathbf{K}' : \mathbf{F}_j'] < \omega$, i.e. $j \in Q_2$. Also, given $\overline{e} \in \mathbf{K}'^{\circ}$ generic over $\overline{a}, \overline{b}, \overline{h}$, we have that $\overline{e}, \overline{e}^{-1} \in \mathbf{F}_j'$ and $(\overline{e}, \overline{a}) \in \mathbf{E}_j$. Thus, $(1, \overline{a}) \in \mathbf{E}_j$, i.e. $j \in Q_1$, a contradiction since $Q_1 \cap Q_2 = \emptyset$.

Therefore, $\neg\underline{\mathbf{E}_j}(\overline{x}, \overline{a}) \in q$ for each j. On the other hand $\underline{\mathbf{D}}(\overline{x}, \overline{a}) \in q$. Indeed, since $\overline{h} \in \mathbf{K}'^{\circ}$, $\overline{h} \cdot \overline{e} \in \mathbf{K}'^{\circ}$ is generic over $\overline{a}, \overline{b}, \overline{h}$. So $(\overline{h}^{-1} \cdot \overline{e}, 1) \in \mathbf{K}$ and $(\overline{h}, \overline{a}) \in \mathbf{D}$. So $(\overline{e}, \overline{a}) \in \mathbf{D}$. Hence, every generic element $\overline{e}$ in $\mathbf{K}'^{\circ}$ over $\overline{a}, \overline{b}, \overline{h}$

is such that $(\overline{e}, \overline{a}) \in \mathbf{D} \setminus \bigcup \mathbf{E}_j \subseteq \mathbf{H}$. So, every generic element in $\mathbf{K}'^{\circ}$ over $\overline{a}, \overline{b}, \overline{h}$ belongs to $\mathbf{H}_{\overline{a}}$. By proposition 3.13, we conclude that $\mathbf{K}'^{\circ} \subseteq \mathbf{H}_{\overline{a}}$. Hence, $\mathbf{K}'^{\circ} \subseteq \mathbf{H}_{\overline{a}}^{\circ}$.

Finally, consider $\mathbf{N} = \mathbf{H}_{\overline{a}} \cap \mathbf{K}'$. We have proved that $[\mathbf{H}_{\overline{a}} : \mathbf{N}] < \omega$ and $[\mathbf{K}' : \mathbf{N}] < \omega$. Since $\mathbf{H}_{\overline{a}}$ is $\overline{a}$-definable and $[\mathbf{H}_{\overline{a}} : \overline{N}] < \omega$, $\mathbf{N}$ has finitely many conjugates over $\overline{a}$. Thus, $\mathbf{N}$ is $\mathrm{acl}(\overline{a})$-definable. By a similar argument, $\mathbf{N}$ is $\mathrm{acl}(\overline{b})$-definable. Let $r = \mathrm{cb}(\mathbf{N})$ in $\mathfrak{C}^{\mathrm{eq}}$, then $r \in \mathrm{acl}^{\mathrm{eq}}(\overline{a}) \cap \mathrm{acl}^{\mathrm{eq}}(\overline{b})$ and $\overline{a} \mathop{\downarrow} \overline{b}$. By the corollary 2.34, $\overline{a} \mathop{\downarrow} \overline{b}, r$. By monotonicity [proposition 2.31], $\overline{a} \mathop{\downarrow} r$. By symmetry [theorem 2.33], $r \mathop{\downarrow} \overline{a}$. Since $\mathrm{MR}(r/\overline{a}) = 0$, we conclude that $\mathrm{MR}(r) = 0$. So $r \in \mathrm{acl}^{\mathrm{eq}}(\emptyset)$. Thus, $\mathbf{N}$ is $\mathrm{acl}(\emptyset)$-definable. $\square$

Lemma 3.21. *Let $\mathbf{H} \leq \mathbf{G}$ be a $\mathrm{acl}(\emptyset)$-definable subgroup and $c \in \mathbf{G}$. Then, any infinite intersection of conjugates of $g\mathbf{H}$ is empty.*

Proof. Indeed, since there is a finite number of conjugates $\mathbf{H}_1, \ldots, \mathbf{H}_k$ of $\mathbf{H}$, for any automorphism $\mathbf{f}$, $\mathbf{f}(c\mathbf{H}) = \mathbf{f}(c)\mathbf{H}_i$ for some $i \in \{1, \ldots, k\}$. Now, for any $c, d \in \mathbf{G}$ and any $i \in \{1, \ldots, k\}$, either $c\mathbf{H}_i \cap d\mathbf{H}_i = \emptyset$ or $c\mathbf{H}_i = d\mathbf{H}_i$. Thus, when $\{\mathbf{f}_n(c\mathbf{H})\}_{n \in \mathbb{N}}$ is infinite, by the pigeonhole principle, there are $i \in \{1, \ldots, k\}$ and $n, m \in \mathbb{N}$ such that $\mathbf{f}_n(c\mathbf{H}) = \mathbf{f}_n(c)\mathbf{H}_i$ and $\mathbf{f}_m(c\mathbf{H}) = \mathbf{f}_m(c)\mathbf{H}_i$ and $\mathbf{f}_n(c)\mathbf{H}_i \cap \mathbf{f}_m(c)\mathbf{H}_i = \emptyset$. $\square$

Theorem 3.22. *Let $\mathfrak{C}$ be totally transcendental. Then, $\mathbf{G}$ is one-based if and only if, for every $n \in \mathbb{N}$, every definable subclass of $\mathbf{G}^n$ is a finite boolean combination of cosets of definable subgroups of $\mathbf{G}^n$.*

Proof. ($\Rightarrow$) By the lemma 3.19, it suffices to prove that for pair of global types $\mathbf{p}, \mathbf{p}' \in \langle \underline{\mathbf{G}^n} \rangle_{\mathbf{S}^{\mathfrak{C}}_n(\mathbf{C})}$ there is a definable group $\mathbf{H} \leq \mathbf{G}^n$ and an element $g \in \mathbf{G}^n$ such that $\mathbf{p} \in \langle \underline{\mathbf{H}g} \rangle$ and $\mathbf{p}' \notin \langle \underline{\mathbf{H}g} \rangle$. Indeed, assume that $\mathrm{MR}(\mathbf{p}) \leq \mathrm{MR}(\mathbf{p}')$. Let A_0 be a finite sorted subset such that $\mathbf{p}$ and $\mathbf{p}'$ do not fork over A_0 and $\mathbf{p}_{|A_0}$ and $\mathbf{p}'_{|A_0}$ are stationary. Let c realize $\mathbf{p}_{|A_0}$ and c' realize $\mathbf{p}'_{|A_0, c}$. By (3) of the theorem 3.18, there is an element $a \in \mathbf{G}^n$ such that $c \in \mathrm{Stab}_{\mathbf{p}} a$. If $c' \in \mathrm{Stab}_{\mathbf{p}} a$, then $c' \cdot c^{-1} \in \mathrm{Stab}_{\mathbf{p}}$. Now, since $c' \cdot c^{-1}$ and c' are interdefinable over A_0, c and $\mathrm{MR}(\mathbf{p}) \geq \mathrm{MR}(\mathrm{Stab}_{\mathbf{p}})$ [proposition 3.8]:

$$\begin{aligned} \mathrm{MR}(c \cdot c^{-1}/A_0, c) =& \mathrm{MR}(c'/A_0, c) = \mathrm{MR}(c'/A_0) = \mathrm{MR}(\mathbf{p}') \geq \\ \geq& \mathrm{MR}(\mathbf{p}) \geq \mathrm{MR}(\mathrm{Stab}_{\mathbf{p}}) \geq \\ \geq& \mathrm{MR}(c' \cdot c^{-1}/A_0) \geq \mathrm{MR}(c' \cdot c^{-1}/A_0, c). \end{aligned}$$

So, $c' \cdot c^{-1} \mathop{\downarrow}_{A_0} c$. Also, $c \mathop{\downarrow}_{A_0} c'$ by symmetry [theorem 2.33], so $c' \cdot c^{-1} \mathop{\downarrow}_{A_0} c'$ since $c' \cdot c^{-1}$ and c are interdefinable over A_0, c'. Thus, $\mathrm{tp}(c/A_0, c' \cdot c^{-1}) = \mathbf{p}_{|A_0, c' \cdot^{-1}}$ and $\mathrm{tp}(c'/A_0, c' \cdot c^{-1}) = \mathbf{p}_{|A_0, c' \cdot c^{-1}}$. Since $c' \cdot c^{-1} \in \mathrm{Stab}_{\mathbf{p}}$

$$\begin{aligned} \mathbf{p}'_{|A_0, c' \cdot c^{-1}} =& \mathrm{tp}(c'/A_0, c' \cdot c^{-1}) = \\ =& \mathrm{tp}((c' \cdot c^{-1}) \cdot c/A_0, c' \cdot c^{-1}) = (c' \cdot c^{-1} \cdot \mathbf{p})_{|A_0, c' \cdot c^{-1}} = \\ =& \mathbf{p}_{|A_0, c' \cdot c^{-1}}. \end{aligned}$$

Hence, $\mathbf{p} = \mathbf{p}'$. So, if $\mathbf{p} \neq \mathbf{p}'$, then $\underline{\mathrm{Stab}_{\mathbf{p}} a} \in \mathbf{p}$ and $\underline{\mathrm{Stab}_{\mathbf{p}} a} \notin \mathbf{p}'$.

So, by the lemma 3.19, every definable subclass of $\mathbf{G}^n$ is a boolean combination of right cosets of $\mathbf{G}^n$. Now, let $\mathbf{H}$ be a definable subgroup of $\mathbf{G}^n$, then $\mathbf{H}^\circ \leq (\mathbf{G}^n)^\circ$. Since $(\mathbf{G}^n)^\circ$ is abelian by (1) of theorem 3.18, $\mathbf{H}^\circ \trianglelefteq (\mathbf{G}^n)^\circ$. And $(\mathbf{G}^n)^\circ \trianglelefteq \mathbf{G}^n$, so $\mathbf{H}^\circ \trianglelefteq \mathbf{G}^n$. Therefore, every right coset is a finite union of left cosets. The latter implies that every definable subclass of $\mathbf{G}^n$ is a boolean combination of cosets.

($\Leftarrow$) By lemma 3.20, every definable subclass of $\mathbf{G}^n$ is a boolean combination of cosets of $\mathrm{acl}(\emptyset)$-definable subgroups of $\mathbf{G}^n$. Let $a \in \mathbf{G}^n$ and A be a finite sorted subset. Let $p = \mathrm{stp}(a/A)$ and $\mathbf{p}$ be its global non-forking extension. By lemma 2.59, we want to prove that $\mathrm{cb}(\mathbf{p}) \in \mathrm{acl}^{\mathrm{eq}}(a)$. Let $\phi \in p$ be such that $\mathfrak{C} \models \forall \overline{x}(\phi \rightarrow \underline{\mathbf{G}^n})$, $\mathrm{MR}(\phi) = \mathrm{MR}(p)$ and $\mathrm{Md}(\phi) = \mathrm{Md}(p)$. By corollary 2.41, we know that $\mathrm{cb}(\mathbf{p}) \in \mathrm{dcl}^{\mathrm{eq}}(\mathrm{cb}(\phi))$. It suffices to prove that $\mathrm{cb}(\phi) \in \mathrm{acl}^{\mathrm{eq}}(a)$. Let $\mathbf{Y} = \phi[\mathfrak{C}]$. By assumption,

$$\mathbf{Y} = \bigcup_i \bigcap_j \mathbf{E}_{ij} \setminus \bigcup_k \mathbf{D}_{ik}$$

where $\mathbf{E}_{ij}, \mathbf{D}_{ik}$ are cosets of $\mathrm{acl}(\emptyset)$-definable groups for each i, j, k. There is an i such that $a \in \bigcap_j \mathbf{E}_{ij} \setminus \bigcup \mathbf{D}_{ik}$. We may assume that $\mathbf{Y} = \bigcap_j \mathbf{E}_j \setminus \bigcup_k \mathbf{D}_k$. On the other hand, the intersection of cosets is empty or a coset. So, we have that $\mathbf{Y} = \mathbf{E} \setminus \bigcup_k \mathbf{D}_k$ were $\mathbf{E}$ and $\mathbf{D}_k$ are cosets of $\mathrm{acl}(\emptyset)$-definable subgroups for each k. Then,

$$\mathbf{Y} = \bigcap_k \mathbf{E} \setminus \mathbf{D}_k.$$

If $\mathrm{MR}(\mathbf{D}_k) < \mathrm{MR}(\mathbf{E})$, then $\mathrm{MR}(\mathbf{E} \setminus \mathbf{D}_k) = \mathrm{MR}(\mathbf{E})$. So, we may assume that $\mathrm{MR}(\mathbf{E}) = \mathrm{MR}(\mathbf{D}_k)$. Therefore, $\mathbf{E} \setminus \mathbf{D}_k$ is a finite union of cosets of the same group that $\mathbf{D}_k$. We may assume that

$$\mathbf{Y} = \bigcap_k \bigcup_i \mathbf{F}_{ki} = \bigcup_i \bigcap_k \mathbf{F}'_{ik}.$$

There is an i such that $a \in \bigcap_k \mathbf{F}'_{ik}$. We may assume that $\mathbf{Y} = \bigcap_k \mathbf{F}'_k$. Now, note that $\mathrm{cb}(\mathbf{Y}) \in \mathrm{dcl}^{\mathrm{eq}}(\{\mathrm{cb}(\mathbf{F}'_k) : k\})$, so it suffices to prove that $\mathrm{cb}(\mathbf{F}'_k) \in \mathrm{acl}^{\mathrm{eq}}(a)$ for each k. Let $\mathbf{F}$ be a coset of an $\mathrm{acl}(\emptyset)$-definable subgroup such that $a \in \mathbf{F}$. Let $\widetilde{\mathbf{F}} = \bigcap \{\mathbf{f}(\mathbf{F}) : \mathbf{f} \text{ aut. and } a \in \mathbf{f}(\mathbf{F})\}$. Of course, $a \in \widetilde{\mathbf{F}}$. In particular $\widetilde{\mathbf{F}} \neq \emptyset$, so $\widetilde{\mathbf{F}}$ is a finite intersection [lemma 3.21]. Therefore, there are finitely many conjugates of $\mathbf{F}$ to which a belongs. Hence, $\mathrm{cb}(\mathbf{F}) \in \mathrm{acl}^{\mathrm{eq}}(a)$. □

Corollary 3.23. *Let $\mathfrak{C}$ be totally transcendental. Then, $\mathbf{G}$ is one-based if and only if, for every $n \in \mathbb{N}$, every definable subclass of $\mathbf{G}^n$ is a finite boolean combination of cosets of definable connected subgroups of $\mathbf{G}^n$.*

3.5 Almost strongly minimal subgroups

Lemma 3.24. *Let $\mathbf{X} \subseteq \mathbf{G}$ be a strongly minimal definable class. Then, there is a definable subclass $\mathbf{X}_0 \subseteq \mathbf{X}$ such that $\mathbf{X} \setminus \mathbf{X}_0$ is finite and $\mathbf{X}_0$ is indecomposable.*

Proof. Let
$$\mathcal{K} = \{\underline{\mathbf{K}} : \mathbf{K} \leq \mathbf{G} \text{ definable and } {}^{\mathbf{X}}/_{\mathbf{K}} \text{ finite}\}$$
and $\mathbf{K}_0 = \bigcap\{\mathbf{K} : \underline{\mathbf{K}} \in \mathcal{K}\}$. By the descending chain condition [corollary 3.4], $\mathbf{K}_0$ is definable and $\mathbf{K}_0 = \bigcap\{\mathbf{K} : \underline{\mathbf{K}} \in \Delta\}$ where $\Delta \subseteq \mathcal{K}$ is finite. Thus, ${}^{\mathbf{X}}/_{\mathbf{K}_0}$ is finite. Then, $\mathbf{X} = \bigcup_{i=1}^{n}(x_i\mathbf{K}_0) \cap \mathbf{X}$ for some $x_1, \ldots, x_n \in \mathbf{X}$. By 2.13, $1 = \mathrm{Md}_1(\mathbf{X}) = \sum_{i=1}^{n} \mathrm{Md}_1((x_i\mathbf{K}_0) \cap \mathbf{X})$. Therefore, there is just one i such that $\mathrm{Md}_1((x_i\mathbf{K}_0) \cap \mathbf{X}) \neq 0$. Assume that $\mathrm{Md}_1((x_1\mathbf{K}_0) \cap \mathbf{X}) = 1$ and $\mathrm{Md}_1((x_j\mathbf{K}_0) \cap \mathbf{X}) = 0$. Then, $(x_j\mathbf{K}_0) \cap \mathbf{X}$ is finite for $j \neq 1$. Let $\mathbf{X}_0 = (x_1\mathbf{K}_0) \cap \mathbf{X}$, then $\mathbf{X} \setminus \mathbf{X}_0$ is finite. I claim that $\mathbf{X}_0$ is indecomposable. Indeed, if $\mathbf{K} \leq \mathbf{G}$ is a definable subgroup, either ${}^{\mathbf{X}}/_{\mathbf{K}}$ is infinite or $\underline{\mathbf{K}} \in \mathcal{K}$. Since $\mathbf{X} \setminus \mathbf{X}_0$ is finite, ${}^{\mathbf{X}_0}/_{\mathbf{K}}$ is infinite when ${}^{\mathbf{X}}/_{\mathbf{K}}$ is so. If $\underline{\mathbf{K}} \in \mathcal{K}$, then $\mathbf{K}_0 \subseteq \mathbf{K}$, so ${}^{\mathbf{X}_0}/_{\mathbf{K}} = \{x_1\mathbf{K}\}$. □

Proposition 3.25. *Let $\mathfrak{C}$ be a monster model over L and assume $\mathbf{G}$ has finite Morley's rank. Let $\mathbf{X} \subseteq \mathbf{G}$ be a strongly minimal definable subclass. Then, there exists a connected definable subgroup $\mathbf{H} \leq \mathbf{G}$ such that $\mathbf{H} \subseteq \mathrm{dcl}(\mathbf{X})$ and ${}^{\mathbf{X}}/_{\mathbf{H}}$ is finite.*

Proof. By lemma 3.24, let $\mathbf{X}_0 \subseteq \mathbf{X}$ be an indecomposable definable subset such that $\mathbf{X} \setminus \mathbf{X}_0$ is finite. Let $a \in \mathbf{X}_0$, write $\mathbf{X}_a := a^{-1}\mathbf{X}_0$. Thus, $\{\underline{\mathbf{X}_a}\}_{a \in \mathbf{X}_0}$ is a family of indecomposable definable subsets such that, for each $a \in \mathbf{X}_0$, $1 \in \mathbf{X}_a$ and every element of $\mathbf{X}_a$ is definable from elements of $\mathbf{X}$. By the indecomposability theorem [theorem 3.16], $\mathbf{H} = \langle \bigcup_{a \in \mathbf{X}_0} \mathbf{X}_a \rangle$ is a connected definable subgroup of $\mathbf{G}$ generated by a finitely many $\mathbf{X}_a$. Thus, every element of $\mathbf{H}$ is definable from finitely many elements of $\mathbf{X}$. On the other hand, $\mathbf{X} = \mathbf{X}_0 \cup \{a_1, \ldots, a_n\}$, so $\mathrm{card}({}^{\mathbf{X}}/_{\mathbf{H}}) \leq \mathrm{card}({}^{\mathbf{X}_0}/_{\mathbf{H}}) + n = 1 + n$ since $b_1^{-1} \cdot b_2 \in \mathbf{X}_{b_1} \subseteq \mathbf{H}$ for any $b_1, b_2 \in \mathbf{X}_0$. □

Assume $\mathbf{G}$ has finite Morley's rank and let $\mathbf{X} \subseteq \mathbf{G}$ be a strongly minimal definable subclass. Consider the set

$$\boldsymbol{\mathcal{B}}_{\boldsymbol{X}} = \left\{ \underline{\mathbf{B}} : \begin{array}{l} \mathbf{B} \leq \mathbf{G} \text{ is definable, connected and there} \\ \text{is a finite sorted subset } F \text{ such that} \mathbf{B} \subseteq \mathrm{acl}(F, \mathbf{X}) \end{array} \right\}.$$

By the proposition 3.25, $\boldsymbol{\mathcal{B}}_{\boldsymbol{X}} \neq \emptyset$. Since $\mathrm{MR}(\mathbf{G}) < \omega$, $\{\mathrm{MR}(\mathbf{B}) : \underline{\mathbf{B}} \in \boldsymbol{\mathcal{B}}_{\boldsymbol{X}}\}$ is a finite set of natural numbers. Therefore, there is $\mathbf{B}_{\mathbf{X}} \in \boldsymbol{\mathcal{B}}_{\boldsymbol{X}}$ such that $\mathrm{MR}(\mathbf{B}_{\mathbf{X}}) = \max\{\mathrm{MR}(\mathbf{B}) : \underline{\mathbf{B}} \in \boldsymbol{\mathcal{B}}_{\boldsymbol{X}}\}$. I claim that every element of $\boldsymbol{\mathcal{B}}_{\boldsymbol{X}}$ is contained in $\mathbf{B}_{\mathbf{X}}$. Indeed, let $\underline{\mathbf{B}_1}, \underline{\mathbf{B}_2} \in \boldsymbol{\mathcal{B}}_{\boldsymbol{X}}$ and F_1, F_2 such that $\mathbf{B}_i \subseteq \mathrm{acl}(F_i, \mathbf{X})$, for $i \in \{1, 2\}$. Then, $\mathbf{B}_1$ and $\mathbf{B}_2$ are indecomposable, so $\mathbf{B}' = \langle \mathbf{B}_1 \cup \mathbf{B}_2 \rangle \leq \mathbf{G}$ is a connected definable subgroup by the indecomposability theorem [theorem 3.16]. Now, $\mathbf{B}' \subseteq \mathrm{dcl}(\mathbf{B}_1 \cup \mathbf{B}_2) \subseteq \mathrm{acl}(F_1 \cup F_2, \mathbf{X})$. Thus, $\mathbf{B}' \in \boldsymbol{\mathcal{B}}_{\boldsymbol{X}}$ and $\mathrm{MR}(\mathbf{B}_1) \leq \mathrm{MR}(\mathbf{B}')$. Hence, for $\mathbf{B}_1 = \mathbf{B}_{\mathbf{X}}$ and $\mathbf{B}_2 = \mathbf{B}$ arbitrary, we conclude that $\mathrm{MR}(\mathbf{B}_{\mathbf{X}}) = \mathrm{MR}(\langle \mathbf{B}_{\mathbf{X}} \cup \mathbf{B} \rangle)$. Since $\langle \mathbf{B}_{\mathbf{X}} \cup \mathbf{B} \rangle$ is connected, $\mathbf{B}_{\mathbf{X}} = \langle \mathbf{B}_{\mathbf{X}} \cup \mathbf{B} \rangle$.

Note that, for any two strongly minimal definable sets $X_1 \subseteq G$ and $X_2 \subseteq G$, if $\mathbf{X}_1 \not\perp \mathbf{X}_2$, by lemma 2.62, $\boldsymbol{\mathcal{B}}_{\mathbf{X}_1} = \boldsymbol{\mathcal{B}}_{\mathbf{X}_2}$ and $\mathbf{B}_{\mathbf{X}_1} = \mathbf{B}_{\mathbf{X}_2}$.

Finally, we define $\boldsymbol{\mathcal{B}_G}$ and $\mathbf{B_G}$ as follows:

$$\boldsymbol{\mathcal{B}_G} = \left\{ \underline{\mathbf{B}} \ : \ \begin{array}{l} \mathbf{B} \leq \mathbf{G} \text{ is definable, connected and such that there} \\ \text{is } \mathbf{X} \subseteq \mathbf{G} \text{ definable with } \mathrm{MR}(\mathbf{X}) = 1 \text{ and a finite} \\ \text{sorted subset } F \text{ such that } \mathbf{B} \subseteq \mathrm{acl}(F, \mathbf{X}) \end{array} \right\}.$$

Note that $\boldsymbol{\mathcal{B}_X} \subseteq \boldsymbol{\mathcal{B}_G}$ for every strongly minimal definable subclass $\mathbf{X} \subseteq \mathbf{G}$. Also, note that both definitions coincide when $\mathbf{G}$ is strongly minimal — in this case, $\boldsymbol{\mathcal{B}_G} = \{\underline{\mathbf{B}} \ : \ \mathbf{B} \leq \mathbf{G} \text{ is definable and connected}\}$. Since $\mathrm{MR}(\mathbf{G}) < \omega$, $\{\mathrm{MR}(\mathbf{B}) \ : \ \underline{\mathbf{B}} \in \boldsymbol{\mathcal{B}_G}\}$ is a finite set of natural numbers. Therefore, there is $\mathbf{B_G} \in \boldsymbol{\mathcal{B}_G}$ such that $\mathrm{MR}(\mathbf{B_G}) = \max\{\mathrm{MR}(\mathbf{B}) \ : \ \underline{\mathbf{B}} \in \boldsymbol{\mathcal{B}_G}\}$. As in the above case, we can prove that $\mathbf{B_G}$ is maximum in $\boldsymbol{\mathcal{B}_G}$.

Proposition 3.26. *Assume $\mathbf{G}$ has finite Morley's rank. Then, there are $\mathbf{X}_1$, $\dots$, $\mathbf{X}_n$, strongly minimal definable subclasses of $\mathbf{G}$, such that $\mathbf{B_G} = \langle\bigcup_{i=1}^n \mathbf{B}_{\mathbf{X}_i}\rangle$. Moreover, there are $\mathbf{X}_1, \dots, \mathbf{X}_n$, strongly minimal definable subclasses of $\mathbf{G}$ pairwise orthogonal such that $\mathbf{B_G} = \langle\bigcup_{i=1}^n \mathbf{B}_{\mathbf{X}_i}\rangle$.*

Proof. Let $\mathbf{Y} \subseteq \mathbf{G}$ definable and F finite sorted subset be such that $\mathrm{MR}(\mathbf{Y}) = 1$ and $\mathbf{B_G} \subseteq \mathrm{acl}(F, \mathbf{Y})$. Let

$$\mathcal{K} = \{\underline{\mathbf{H}} \ : \ \mathbf{H} = \langle \mathbf{B}_{\mathbf{X}_1} \cup \cdots \cup \mathbf{B}_{\mathbf{X}_n}\rangle \text{ where } \mathbf{X}_1, \dots, \mathbf{X}_n \subseteq \mathbf{G} \text{ strongly minimal}\}.$$

Since $\mathrm{MR}(\mathbf{G}) < \omega$, $\{\mathrm{MR}(\mathbf{B}) : \underline{\mathbf{B}} \in \mathcal{K}\}$ is a finite set of natural numbers. Let $\underline{\mathbf{H}} \in \mathcal{K}$ be such that $\mathrm{MR}(\mathbf{H}) = \max\{\mathrm{MR}(\mathbf{B}) \ : \ \mathbf{B} \in \mathcal{K}\}$. Let $\mathbf{H} = \langle \mathbf{B}_{\mathbf{X}_1} \cup \ldots \cup \mathbf{B}_{\mathbf{X}_n}\rangle$. Let A be a finite sorted subset such that $F \subseteq A$ and $\mathbf{Y}$, $\mathbf{X}_1, \dots, \mathbf{X}_n$, $\mathbf{B_G}$ and $\mathbf{H}$ are A-definable. It is clear that $\mathbf{H} \subseteq \mathbf{B_G}$. Suppose that $\mathbf{B_G} \setminus \mathbf{H} \neq \emptyset$. Since $\mathbf{B_G}$ is connected, $\mathbf{B_G} \setminus \mathbf{H} \neq \emptyset$ implies that $\mathrm{MR}(\mathbf{H}) < \mathrm{MR}(\mathbf{B_G})$. Then, $[\mathbf{B_G} : \mathbf{H}] \geq \omega$ by the descending chain condition. So, ${}^{\mathbf{B_G}}/_{\mathbf{H}}$ is a proper definable class of $\mathfrak{C}^{\mathrm{eq}}$. Then, there is an imaginary element $\widetilde{c} = c\mathbf{H} \in {}^{\mathbf{B_G}}/_{\mathbf{H}} \subseteq \mathbf{C}^{\mathrm{eq}}$ non-algebraic over A. Since $c \in \mathbf{B_G}$, there is a finite Y_0 from $\mathbf{Y}$ such that c is algebraic over Y_0, A. Thus, $\widetilde{c}$ is algebraic over Y_0, A. We may assume that $\widetilde{c}$ is not algebraic over Y', A for any proper subset $Y' \subset Y_0$. Let $y \in Y_0$ and set $Y_1 = Y_0 \setminus \{y\}$. Then, $\widetilde{c} \in \mathrm{acl}^{\mathrm{eq}}(A, Y_0) \setminus \mathrm{acl}^{\mathrm{eq}}(A, Y_1)$. Thus, $\widetilde{c} \not\perp_{A,Y_1} y$. Since $\mathrm{MR}(y/A) = 1$, by symmetry [theorem 2.33], we have that $y \in \mathrm{acl}^{\mathrm{eq}}(\widetilde{c} \cup Y_1, A)$. So $c \in \mathrm{acl}(A, Y_1 \cup \{y\}) \subseteq \mathrm{acl}^{\mathrm{eq}}(A, Y_1, \widetilde{c})$ and $c \notin \mathrm{acl}(A, Y_1)$. Thus, by theorem 2.23, $0 < \mathrm{MR}(c/A, Y_1) \leq \mathrm{MR}(y/A, Y_1) = 1$, so $\mathrm{MR}(c/A, Y_1) = 1$. Also, $\mathrm{MR}(\widetilde{c}/A, Y_1) = 1$ by the same theorem, since c and $\widetilde{c}$ are interalgebraic over A, Y_1. So, since $\mathrm{MR}(c/A, Y_1) = 1$, there is a A, Y_1-definable class $\mathbf{T} \subseteq \mathbf{B_G}$ such that $c \in \mathbf{T}$ and $\mathrm{MR}(\mathbf{T}) = 1$. Since $c \in \mathrm{acl}^{\mathrm{eq}}(A, Y_1, \widetilde{c})$, there is a formula $\phi(\overline{x}, y) \in \mathrm{For}_{\overline{x},y} L^{\mathrm{eq}}(A, Y_1)$ such that $\mathfrak{C} \models \phi[c, \widetilde{c}]$ and $\mathrm{card}(\phi(\overline{x}, \widetilde{c}')[\mathfrak{C}^{\mathrm{eq}}]) = \mathrm{card}(\phi(\overline{x}, \widetilde{c})[\mathfrak{C}^{\mathrm{eq}}])$ for every $\widetilde{c}'$. Let $\psi(\overline{x}) = \phi(\overline{x}, \pi_{\mathbf{H}}(\overline{x}))$. Thence, $\psi \in \mathrm{tp}(c/A, Y_1)$, so $\mathbf{T}' = \mathbf{T} \cap \psi[\mathfrak{C}^{\mathrm{eq}}]$ is such that $c \in \mathbf{T}'$. Of course, $\mathbf{T}'$ is such that $\mathrm{MR}(\mathbf{T}') \geq 1$. Now, every element of $d \in \mathbf{T}'$ is algebraic over $d\mathbf{H}$ and A, Y_1. Thus, $[\mathbf{T}' : \mathbf{H}]$ must be infinite. Let us prove that for any $\mathbf{X} \subseteq \mathbf{G}$ such that $\mathrm{MR}(\mathbf{X}) = 1$, ${}^{\mathbf{X}}/_{\mathbf{H}}$ is finite. By proposition 2.12, it suffices to prove that ${}^{\mathbf{X}}/_{\mathbf{H}}$ is finite for every strongly minimal definable class $\mathbf{X} \subseteq \mathbf{G}$. The latter is clear, since $\mathbf{B_X} \subseteq \mathbf{H}$ by maximality of $\mathbf{H}$ and ${}^{\mathbf{X}}/_{\mathbf{B_X}}$ is finite by proposition 3.25. Therefore, $\mathbf{H} = \mathbf{B_G}$. Finally, since $\boldsymbol{\mathcal{B}_X} = \boldsymbol{\mathcal{B}_{X'}}$ if $\mathbf{X} \not\perp \mathbf{X}'$, we may assume that $\mathbf{X}_1, \dots, \mathbf{X}_n$ are pairwise orthogonal. $\square$

3.6 Orthogonality and groups

Lemma 3.27. *Let $\mathfrak{C}$ be totally transcendental, $\mathbf{E}$ a 0-definable class and g algebraic from the coordinates of $\mathbf{E}$. Then, there is an imaginary element $f \in \operatorname{dcl}^{\mathrm{eq}}(g) \cap \operatorname{dcl}^{\mathrm{eq}}(\mathbf{E})$ such that $g \in \operatorname{acl}^{\mathrm{eq}}(f)$.*

Proof. Let $\phi(x, \overline{e})[\mathfrak{C}]$ be a finite $\overline{e}$-definable set such that $\mathfrak{C} \models \phi(g, \overline{e})$ and $\overline{e} \in \mathbf{E}^n$. Let $f = \operatorname{cb}(\phi(g, \overline{y})[\mathfrak{C}] \cap \mathbf{E}^n)$. Since $\mathbf{E}$ is 0-definable, $f \in \operatorname{dcl}^{\mathrm{eq}}(g)$ by theorem 1.29. It follows that $f \in \operatorname{dcl}^{\mathrm{eq}}(\mathbf{E})$ by corollary 2.30.

Let us prove that $g \in \operatorname{acl}^{\mathrm{eq}}(f)$. Let $\psi(w, \overline{y})$ be an L^{eq}-formula be such that $\psi(f, \overline{y})[\mathfrak{C}] = \phi(g, \overline{y})[\mathfrak{C}] \cap \mathbf{E}^n$. Let $k = \operatorname{card}(\phi(x, \overline{e})[\mathfrak{C}])$, we prove that there are at most k-conjugates of g over f. Indeed, let $g_0, \ldots, g_k$ be conjugates of g over f. We know that $\operatorname{tp}(g_i/f) = \operatorname{tp}(g/f)$ for each i. Since $\forall \overline{y}\, (\psi(f, \overline{y}) \leftrightarrow \phi(x, \overline{y}) \wedge \underline{\mathbf{E}}(\overline{y})) \in \operatorname{tp}(g/f)$, we conclude that $\phi(g_i, \overline{y})[\mathfrak{C}] \cap \mathbf{E}^n = \phi(g_i, \overline{y})[\mathfrak{C}] \cap \mathbf{E}^n$ for each i, j. So $g_0, \ldots, g_n \in \phi(x, \overline{e})[\mathfrak{C}]$ and by the pigeonhole principle $g_i = g_j$ for some $i \neq j$. □

Theorem 3.28. *Let $\mathfrak{C}$ be totally transcendental, $\mathbf{D}$ a strongly minimal 0-definable subclass and $\mathbf{E}$ a 0-definable class. Assume $\mathbf{G}$ is almost strongly minimal respect to $\mathbf{D}$. Then, $\mathbf{G} \not\perp \mathbf{E}$ if and only if there are $(\mathbf{H}, *)$, a definable group of $\mathfrak{C}^{\mathrm{eq}}$, and a definable onto homomorphism $\mathbf{h} : \mathbf{G} \to \mathbf{H}$ such that $\ker(\mathbf{h})$ is finite and $\mathbf{H} \subseteq \operatorname{dcl}^{\mathrm{eq}}(\mathbf{E})$.*

Proof. ($\Leftarrow$) Let $A \subseteq \mathbf{C}^{\mathrm{eq}}$ be a finite sorted subset such that $\mathbf{H}$ and $\mathbf{h}$ are A-definable. Since g is algebraic over $\mathbf{h}(g)$ for every $g \in \mathbf{G}$, we have that $g \not\mathrel{\perp\!\!\!\perp}_A \mathbf{h}(g)$ for any g not algebraic over $\operatorname{acl}^{\mathrm{eq}}(A)$. So, $\mathbf{G} \not\perp \mathbf{H}$. Then, by corollary 2.64, there is a finite sorted set A such that $\mathbf{G} \subseteq \operatorname{acl}^{\mathrm{eq}}(\mathbf{H}, A) \subseteq \operatorname{acl}^{\mathrm{eq}}(\mathbf{E}, A)$. So $\mathbf{G} \not\perp \mathbf{E}$.

($\Rightarrow$) First, assume that $\mathbf{G}$ is connected. Since $\mathbf{G} \not\perp \mathbf{E}$, by corollary 2.64, there is a finite sorted subset A such that $\mathbf{G} \subseteq \operatorname{acl}^{\mathrm{eq}}(\mathbf{E}, A)$. In particular, $\mathbf{E}$ is a proper class. Adding A to the language, assume that A is empty.

Step 1 (First approximation to h). There is a definable function $\mathbf{h}_2 : \mathbf{G} \to \operatorname{dcl}^{\mathrm{eq}}(\mathbf{E})$ such that $\mathbf{h}_2^{-1}(\{f\})$ is finite for every f.

Then, for every $g \in \mathbf{G}$ there is $f_g \in \operatorname{dcl}^{\mathrm{eq}}(g) \cap \operatorname{dcl}^{\mathrm{eq}}(\mathbf{E})$ such that $g \in \operatorname{acl}^{\mathrm{eq}}(f_g)$ by lemma 3.27, since $\mathbf{G} \subseteq \operatorname{acl}^{\mathrm{eq}}(\mathbf{E})$. Let $\psi_g(\overline{y}, \overline{e})$ be an $L(\mathbf{E})$-formula defining f_g where $\overline{e} \in \mathbf{E}^{n_g}$ and $\varphi_g(\overline{x}, \overline{y})$ a formula such that $\varphi(g, \overline{y})[\mathfrak{C}] = \{g\}$ and $\varphi(\overline{x}, f_g)[\mathfrak{C}]$ is finite of cardinality k_g. We may assume that $\varphi_g(\overline{x}, f)[\mathfrak{C}] \subseteq \mathbf{G}$ is finite of cardinality k_g for every f, that $\varphi_g(g', \overline{y})[\mathfrak{C}]$ is a singleton for any $g' \in \mathbf{G}$ and that $\psi_g(\overline{y}, \overline{e}')[\mathfrak{C}]$ is a singleton for every $\overline{e}' \in \mathbf{E}^{n_g}$. Consider

$$\phi_g(\overline{x}) = \exists \overline{y}, \exists \overline{z}\, (\varphi_g(\overline{x}, \overline{y}) \wedge \psi_g(\overline{y}, \overline{z}) \wedge \underline{\mathbf{E}^{n_g}}(\overline{z}))\,.$$

We have chosen φ_g such that $\operatorname{tp}(g) \in \langle \phi_g \rangle \subseteq \langle \underline{\mathbf{G}} \rangle$. Thus,

$$\bigcup_{g \in \mathbf{G}} \langle \phi_g \rangle = \langle \underline{\mathbf{G}} \rangle.$$

By compactness of $\mathbf{S}^{\mathfrak{C}}_{\overline{x}}$ [Theorem 1.18], there are finitely many $g_1, \dots, g_n \in \mathbf{G}$ such that $\bigcup_{i=1}^{n} \langle \phi_{g_i} \rangle = \langle \underline{\mathbf{G}} \rangle$. Consider $\phi'_i = \phi_i \wedge \bigwedge_{j<i} \neg \phi_j$. Thus, $\phi'_1[\mathfrak{C}], \dots, \phi'_n[\mathfrak{C}]$ is a partition of $\mathbf{G}$. We may assume that $\varphi_1(\overline{x}, \overline{y}), \dots, \varphi_n(\overline{x}, \overline{y})$ have the same free variables — use definable elements of $\mathrm{dcl}^{\mathrm{eq}}(\mathbf{E})$. Define $\mathbf{h}_2(g)$ such that $\mathfrak{C} \models \varphi_i[g, \mathbf{h}_2(g)]$ when $\mathfrak{C} \models \phi_i[g]$. Then, $\mathbf{h}_2 : \ \mathbf{G} \to \mathrm{dcl}^{\mathrm{eq}}(\mathbf{E})$ is a definable function. Indeed, if $\mathfrak{C} \models \phi'_i[g, \mathbf{h}_2(g)]$, then $\{\mathbf{h}(g)\} = \psi_{g_i}(\overline{y}, \overline{e'})[\mathfrak{C}]$ for some $\overline{e}' \in \mathbf{E}^{n_{g_i}}$, so $\mathbf{h}(g) \in \mathrm{dcl}^{\mathrm{eq}}(\mathbf{E})$. Also, $\mathbf{h}_2^{-1}(f) \subseteq \bigcup_{i=1}^{n} \varphi_i(\overline{x}, f)[\mathfrak{C}]$ is finite.

Step 2 (Kernel) Since $\mathbf{G}$ is connected, it has a unique global generic type. Let $\mathbf{p}$ be the global generic type in $\mathbf{G}$. Let

$$\begin{aligned} K =& \{b \in \mathbf{G} \ : \ \mathbf{h}_2(b \cdot \overline{x}) = \mathbf{h}_2(\overline{x}) \in \mathbf{p}\} = \\ =& \{b \in \mathbf{G} \ : \ \mathbf{h}_2(b \cdot g) = \mathbf{h}_2(g) \text{ for one (all) } g \in \mathbf{G} \text{ generic over } b\}. \end{aligned}$$

I claim that K is a finite A-definable normal subgroup of $\mathbf{G}$. Adding A to the language, assume that A is empty.
i. By the theorem 2.28, K is 0-definable.
ii. If K is infinite, let $\{b_i\}_{i\in\omega} \subseteq K$ and $g \in \mathbf{G}$ generic over $\{b_i\}_{i\in\omega}$, then $\{b_i \cdot g\}_{i\in\omega} \subseteq \mathbf{h}_2^{-1}(g)$ which is finite. So, K is a finite definable set.
iii. It is clear that $0 \in K$. On the other hand, if $b \in K$, let g be generic in $\mathbf{G}$ over b, by theorem 2.23, $b \cdot g$ is generic in $\mathbf{G}$ over b^{-1} and $\mathbf{h}_2(b^{-1} \cdot (b \cdot g)) = \mathbf{h}_2(g) = \mathbf{h}_2(b \cdot g)$. Also, if $a, b \in K$, let g be generic in $\mathbf{G}$ over a, b, by theorem 2.23, $b \cdot g$ is generic in $\mathbf{G}$ over a. Therefore, $\mathbf{h}_2((a \cdot b) \cdot g) = \mathbf{h}_2(a \cdot (b \cdot g)) = \mathbf{h}_2(b \cdot g) = \mathbf{h}_2(g)$, so $a \cdot b \in K$. Hence, K is a subgroup.
iv. Let $\psi(\overline{x}, \overline{y})$ be the formula given by $\mathbf{h}_2(\overline{y} \cdot \overline{x}) = \mathbf{h}_2(\overline{x})$. Then, if $b \in K$, $\psi(\overline{x}, b) \in \mathbf{p}$. Let $a \in \mathbf{G}$, since the conjunction for a is a definable bijection, by corollary 2.25 and unity of the generic type, $\psi(\overline{x}, a \cdot b \cdot a^{-1}) \in \mathbf{p}$. So $a \cdot b \cdot a^{-1} \in K$. Hence, $a \cdot K \cdot a^{-1} = K$ and we conclude that K is normal.

Step 3 (Second approximation to h) Since $\mathbf{G}$ is almost strongly minimal respect to $\mathbf{D}$, $\mathrm{MR}(\mathbf{G}) = r \in \mathbb{N}$ [corollary 2.54]. Let $g_0, \dots, g_{2r}$ be generic elements in $\mathbf{G}$ over A such that $g_i \mathop{\smile\!\!\!\!\!|}\, g_{i+1}, \dots, g_{2r}$. We define $\mathbf{h}_1 : \ \mathbf{G} \to \mathrm{dcl}^{\mathrm{eq}}(\mathbf{E})$ by $\mathbf{h}_1(b) = (\mathbf{h}_2(b \cdot g_0), \dots, \mathbf{h}_2(b \cdot g_{2r}))$. This one is an A-definable function. We prove that $\mathbf{h}_1(a) = \mathbf{h}_2(b) \Rightarrow Ka = Kb$. Indeed, by Lascar's equation [Corollary 2.56], $\mathrm{MR}(a, b) = \mathrm{MR}(a/b) + \mathrm{MR}(b) \leq \mathrm{MR}(a) + \mathrm{MR}(b) \leq 2r$. So,

$$0 \leq \mathrm{MR}(a, b/g_0, \dots, g_{2r}) \leq \mathrm{MR}(a, b/g_1, \dots, g_{2r}) \leq \cdots \leq \mathrm{MR}(a, b) \leq 2r.$$

So, $a, b \mathop{\smile\!\!\!\!\!|}_{g_{i+1},\dots,g_{2r}} g_i$ for some $i \in \{0, \dots, 2r\}$ by the pigeonhole principle. Since $g_i \mathop{\smile\!\!\!\!\!|}\, g_{i+1}, \dots, g_{2r}$, by transitivity [Proposition 2.31] and symmetry [Theorem 2.33], $a, b \mathop{\smile\!\!\!\!\!|}\, g_i$. So, there is an $i \in \{0, \dots, 2r\}$ such that g_i is generic over a, b. Then, $a \cdot g_i$ is generic over $b \cdot a^{-1}$ [Theorem 2.23]. Therefore,

$$\begin{aligned} \mathbf{h}_1(a) = \mathbf{h}_1(b) \Rightarrow & \mathbf{h}_2(a \cdot g_i) = \mathbf{h}_2(b \cdot g_i) = \mathbf{h}_2(b \cdot a^{-1} \cdot (a \cdot g_i)) \\ \Rightarrow & b \cdot a^{-1} \in K \Rightarrow Kb = Ka. \end{aligned}$$

Step 4 (Definition of h and H) We define $\mathbf{h}(g)$ as the class of $(\mathbf{h}'(b\cdot g))_{b\in K}$ under the definable equivalence relation with infinitely many classes $(m_b)_{b\in K} \sim (m'_b)_{b\in K} \Leftrightarrow \{m_b\}_{b\in K} = \{m'_b\}_{b\in K}$. Thus, $\mathbf{h} : \mathbf{G} \to \mathrm{dcl}^{\mathrm{eq}}(\mathbf{E})$ is a A-definable function. Let us prove that $\mathbf{h}(g) = \mathbf{h}(g') \Leftrightarrow Kg = Kg'$. Indeed, if $\mathbf{h}(g) = \mathbf{h}(g')$, there are $b, b' \in K$ such that $\mathbf{h}_1(b\cdot g) = \mathbf{h}_1(b'\cdot g')$, so $Kbg = Kb'g'$, i.e., $Kg = Kg'$. On the other hand, if $Kg = Kg'$, then $\mathbf{h}_1(Kg) = \mathbf{h}_1(Kg')$, i.e., $\mathbf{h}(g) = \mathbf{h}(g')$. Now, we define $\mathbf{H} = \mathrm{Im}\ \mathbf{h} \subseteq \mathbf{E}$ and $*$ via $\mathbf{h}$ by $f * f' = \mathbf{h}(g\cdot g')$ for $g, g' \in \mathbf{G}$ such that $\mathbf{h}(g) = f$ and $\mathbf{h}(g') = g'$. Note that $(\mathbf{H}, *)$ is a group since $\mathbf{h}(g) = \mathbf{h}(g') \Leftrightarrow Kg = Kg'$ and K is a normal subgroup. Thus, $\mathbf{h}$ is an onto definable homomorphism with kernel K finite. Note that $\mathbf{H} = {}^{\mathbf{G}}/_{K}$.

Step 5 (When G is not connected) If $\mathbf{G}$ is not connected, consider the connected component $\mathbf{G}^\circ$. Note that $\mathbf{G} \not\angle \mathbf{E}$ implies $\mathbf{G}^\circ \not\angle \mathbf{E}$. Let $\mathbf{h}^\circ : \mathbf{G}^\circ \to \mathbf{H}^\circ$ be the onto definable homomorphism of finite kernel K° for $\mathbf{G}^\circ$ which we obtained by the steps 1 to 4. Note that $K^\circ \subseteq Z(\mathbf{G}^\circ)$, i.e., $Z(K^\circ) = \mathbf{G}^\circ$. Indeed, this is clear since $\mathbf{G}^\circ$ is connected and $Z(K^\circ)$ has finite index because K° is finite. On the other hand, since $[\mathbf{G} : \mathbf{G}^\circ] < \omega$, let $a_1, \dots, a_m \in \mathbf{G}$ be such that $\{a_1\mathbf{G}^\circ, \dots, a_m\mathbf{G}^\circ\} = {}^{\mathbf{G}}/_{\mathbf{G}^\circ}$. Since $\mathbf{G}^\circ$ is a characteristic subgroup of $\mathbf{G}$ and $Z(\mathbf{G}^\circ)$ is a characteristic subgroup of $\mathbf{G}^\circ$, the conjugates $a_1K^\circ a_1^{-1}, \dots, a_mK^\circ a_m^{-1}$ are subgroups of $Z(\mathbf{G}^\circ)$. Thus, the product $K = a_1K^\circ a_1^{-1} \cdots a_mK^\circ a_m^{-1}$ is a subgroup. Of course, K is a finite definable normal subgroup of $\mathbf{G}$. Let $\widetilde{\mathbf{h}^\circ}(g) = [\mathbf{h}^\circ(g)]_{\mathbf{h}^\circ(K^\circ)} \in \widetilde{\mathbf{H}^\circ} = {}^{\mathbf{H}^\circ}/_{\mathbf{h}^\circ(K^\circ)}$. Let $e_1, \dots, e_m \in \mathrm{dcl}^{\mathrm{eq}}(\mathbf{E})$ be m different elements of the same sort and $\mathbf{H} = \{e_1, \dots, e_m\} \times \widetilde{\mathbf{H}^\circ}$. Define $\mathbf{h}(a_i \cdot g) = (e_i, \widetilde{\mathbf{h}^\circ}(g))$ for $g \in \mathbf{G}^\circ$ and $i \in \{1, \dots, m\}$. Then, $\mathbf{h}(g) = \mathbf{h}(g') \Leftrightarrow gK = g'K$. So, $(\mathbf{H}, *)$ is a definable group defining $*$ in $\mathbf{H}$ by $f * f' = \mathbf{h}(g \cdot g')$ where $g, g' \in \mathbf{G}$, $\mathbf{h}(g) = f$ and $\mathbf{h}(g') = f'$. Then, $\mathbf{h}$ is an onto definable homomorphism with kernel K finite. $\square$

4 Model theory of algebraically closed fields

In this chapter we study some model-theoretic properties of algebraically closed fields applying the results proved in the previous chapters. In the first section we prove the basic properties of the theory of algebraically closed fields, like elimination of imaginaries [Theorem 4.6] or ω-stability [Theorem 4.4]. The most significant result is the equivalence between Morley's rank and Krull's dimension [Theorem 4.13]. Next, we define the basic algebraic-geometric concept of abstract variety, check that are definable in the model-theoretic sense and prove the rigidity theorem [Theorem 4.16]. At the end of the chapter, we apply the results already studied to algebraic groups and abelian varieties. The most relevant results are lemma 4.17, proposition 4.18 and theorem 4.22.

We introduce some notations: Write L_r for the language of rings, i.e., $L_r = \{0, 1, -, +, \cdot\}$. Write ACF for the L_r-theory of algebraically closed fields, i.e., the axioms of fields together with the following sentences

$$\forall y_0, \ldots, y_{n-1} \exists x\, x^n + y_{n-1} \cdot x^{n-1} + \cdots + y_1 \cdot x + y_0 = 0 \qquad \text{for } n \geq 2.$$

Write ACF_p for the L_r-theory of algebraically closed fields with characteristic p ($p = 0$ or prime), i.e.,

$$\begin{aligned} \mathrm{ACF}_p &= \mathrm{ACF} \cup \{1 + \overset{p}{\cdots} + 1 = 0\} && p \text{ prime characteristic} \\ \mathrm{ACF}_0 &= \mathrm{ACF} \cup \{1 + \overset{n}{\cdots} + \neq 0\}_{n \in \mathbb{N}^*} && 0 \text{ characteristic.} \end{aligned}$$

Notation. In the rest of this chapter and except otherwise stated, K will denote an $\aleph_0$-saturated algebraically closed field.

Write $\mathbf{V}_n$ and $\mathbf{I}_n$ for the functions given by

$$\mathbf{V}_n(\Delta) = \{\overline{x} \in K^n \, : \, \forall P \in \Delta \; P(\overline{a}) = 0\}$$

$$\mathbf{I}_n(A) = \{P \in K[x_1, \ldots, x_n] \, : \, \forall \overline{a} \in A \; P(\overline{a}) = 0\}$$

Remember that, by definition, $\mathbf{V}(\Delta)$ is a Zariski closed set and $\mathbf{I}(A)$ is an ideal. Also, remember the Nullstellensatz, i.e., $\mathbf{I}(\mathbf{V}(\Delta)) = \sqrt{\langle \Delta \rangle}$, and Hilbert's basis theorem, i.e., $K[x_1, \ldots, x_n]$ is a noetherian ring.

4.1 Basic model theory of algebraically closed fields

Our first theorem is a basic result of any course of model theory.

Theorem 4.1. ***(Quantifier elimination of*** ACF ***and completeness of*** ACF_p***)***

1. ACF *has quantifier elimination;*

2. ACF_p *is κ-categorical for every every $\kappa > \aleph_0$ and every p prime or* 0.

As a consequence we have the following results:

Corollary 4.2. *The definable sets of algebraically closed fields are the Zariski relative open subsets of Zariski closed sets.*

Proof. By quantifier elimination 4.1 and the basis Hilbert's theorem. □

Corollary 4.3. *Algebraically closed fields are strongly minimal.*

Corollary 4.4. ACF_p *is ω-stable for each p prime or* 0.

Proof. Since algebraically closed fields are strongly minimal, ACF_p is totally transcendental. Therefore, ACF_p is ω-stable by theorem 2.22, since L_r is finite.

Remark. Thus, ACF_p has saturated models [Theorem 2.2].

Proposition 4.5. *Algebraically closed fields with transcendent degree over their prime subfield greater than 3 are not locally modular.*

Proof. Let $K \models \mathrm{ACF}$ be a saturated model. Let $a, b \in K$ be transcendent elements over $\emptyset$ such that $a \mathop{\smile\!\!\!\!\!\!\mid} b$. Let $V = \{(x, y) \; : \; y = ax + b\}$. Since $x \mapsto (x, ax + b)$ is a definable bijection, $\mathrm{MR}(V) = \mathrm{MR}(K) = 1$ and $\mathrm{Md}(V) = \mathrm{Md}(K) = 1$ [Corollary 2.25]. So, V is strongly minimal. Let $p(x, y)$ be the global generic type in V, so $\mathrm{MR}(p) = 1$ and $\mathrm{Md}(p) = 1$. I claim that $\mathrm{cb}(p) = (a, b)$. Indeed, let f be an automorphism leaving p invariant. Then, $(f(a)x + f(b) = y) \in p$. Let (c, d) and (c', d') be two different generic elements of V over $a, b, f(a), f(b)$, then we have the equations

$$\begin{aligned} d &= f(a)c + f(b), \quad & d &= ac + b \\ d' &= f(a)c' + f(b), \quad & d' &= ac' + b \end{aligned}$$

Note that $(c, d) \neq (c', d')$ implies $c \neq c'$. Therefore, $a = (c - c')^{-1} \cdot (d - d') = f(a)$ and $b = f(b)$ too, i.e., f fixes (a, b).

Now, p is strongly minimal and $\mathrm{MR}(\mathrm{cb}(p)) = 2$, so K is not locally modular [Theorem 2.60]. □

Remark. Any regular function is definable. Indeed, if $f : Y \to K$ is regular, there are $U_1, \ldots, U_m$ Zariski open, hence definable, and $g_1, h_1, \ldots, g_m, h_m \in K[x_1 \ldots, x_n]$, hence definable, such that $f_{|U_i}(\overline{x}) = {}^{g_i(\overline{x})}/_{h_i(\overline{x})}$ where $Y \subseteq U_1 \cup \cdots \cup U_m$ and h_i has not zeros in U_i. Therefore, regular functions and morphisms are definable.

Theorem 4.6. ACF_p *has elimination of imaginaries for every $p \geq 0$.*

Proof. By the theorem 1.39 ACF_p has weak elimination of imaginaries. So, by the theorem 1.38, it is suffices to prove that ACF_p eliminates finite imaginaries. Let $D = \{\overline{c}^1, \ldots, \overline{c}^k\}$ be a finite set of n-tuples in the monster model $\mathbf{K}$ and consider the polynomial

$$P(x, y_1, \ldots, y_n) = \prod_{i=1}^{k} (x - \sum_{j=1}^{n} c_j^i \cdot y_j).$$

An automorphism fixes P if and only if it permutes the elements of D. So, the coefficients of P are a canonical base of D. □

We have two notions of algebraic. Now, we prove that the concept of algebraic in the model-theoretic sense and in the field-theoretic sense concide.

Theorem 4.7. ***(Algebraic in model theory and field theory)*** *Let $A \subseteq K$, $a \in K$ and $k \subseteq K$ be the subfield generated by A. Then, $a \in \operatorname{acl}(A)$ if and only if $k(a)/k$ is an algebraic extension.*

Proof. If a is algebraic over k, let $P(x) \in k[x]$ be the minimal polynomial of a. Then, the $L_r(k)$-formula $P(x) = 0$ defines a finite set in which a is. This set is definable over A since every automorphism in the monster extension fixing A fixes k, so leaves the zero set of P invariant.

Assume a is transcendental over k. All transcendental elements over k have the same type over k by quantifier elimination [Theorem 4.1]. Indeed, this is the type generated by the set of $L_r(k)$-formulas $P(x) \neq 0$ for each $P(x) \in k[x]$. On the other hand, we know that there are infinitely many transcendent elements over k by saturation. That implies that every definable set over k in which a is has infinitely many elements. Therefore, since every definable set over A is definable over k, we conclude that $a \notin \operatorname{acl}(A)$. $\square$

However, A-definable in model theory does not mean defined over A in field theory. Write V/k for defined in field-theoretic sense and k-definable for model theory.

Lemma 4.8. ***(Definable in model theory and field theory)*** *Let $A \subseteq K$ and k be the field generated by A. Then, $a \in \operatorname{dcl}(A)$ if and only if $a \in k_{\mathrm{perf}}$ — where k_{perf} is the perfect closure of k in K, i.e., $k_{\mathrm{perf}} = \bigcup_n \mathrm{Fr}^{-n}(k)$ where $\mathrm{Fr}:\ x \mapsto x^p$ is the Frobenius' automorphism of K.*

Proof. If $a \in k_{\mathrm{perf}}$, we can define a by the $L(A)$-formula $x^{n \cdot p} = t^K[\vartheta]$ for $n \in \mathbb{Z}$, where $t^K[\vartheta] \in k$ with ϑ evaluation in A. On the other hand, if $a \in \operatorname{dcl}(A)$, in particular, $a \in \operatorname{dcl}(k_{\mathrm{perf}})$. So a is algebraic over k_{perf}. Let $P(x) \in k_{\mathrm{perf}}[x]$ be the minimal irreducible and separable polynomial of a. If $P(x)$ has degree greater than 1, a is algebraic but no definable. Indeed, there is an automorphism which maps a to other zero of $P(x)$. So, $P(x)$ has degree less or equal than one. Since $P(a) = 0$, $P(x) = x - a$, so $a \in k_{\mathrm{perf}}$. $\square$

Lemma 4.9. ***(Minimal set of definition)*** *Let $V \subseteq K^n$ be a Zariski closed set. Then, there is a subfield $k_0 \subseteq K$ such that V/k_0 and, for any $f \in \operatorname{Aut}(K)$, f leaves V invariant if and only if f fixes k_0.*

Proof. Let $R = {}^{K[x_1,\ldots,x_n]}/_{\mathbf{I}(V)}$, $\mathcal{B} = \{e_0, \ldots, e_m\} \subseteq \{\overline{x}^r \ :\ r \in \mathbb{N}^n\}$ such that $\{[e_0], \ldots, [e_m]\}$ is a basis of R as K-vector space and k_0 be the subfield of K generated by

$$\bigcup_{i=0}^{m} \left\{ \lambda_i \in K \ :\ \exists r \in \mathbb{N}^n \exists \lambda_0, \ldots, \lambda_{i-1}, \lambda_{i+1}, \ldots, \lambda_m \in K \ [\overline{x}^r] = \sum_{j=0}^{m} \lambda_j \cdot [e_j] \right\}.$$

I claim that V/k_0 and that, for every $f \in \operatorname{Aut}(K)$, f leaves V invariant if and only if f fixes k_0.

First we show that $\mathbf{I}(V) = \langle \mathbf{I}(V) \cap k_0[x_1, \ldots, x_n] \rangle_{K[x_1,\ldots,x_n]}$, to prove that V/k_0. Let $P \in \mathbf{I}(V)$. we have that

$$P = \sum_{\alpha \in I} a_\alpha \cdot \overline{x}^\alpha = \sum_{\alpha \in I} a_\alpha \cdot \sum_{j=0}^{m} \lambda(\alpha)_j \cdot e_j + \sum_{\alpha \in I} a_\alpha \cdot \left(\overline{x}^\alpha - \sum_{j=0}^{m} \lambda(\alpha)_j \cdot e_j \right).$$

Since $P \in \mathbf{I}(V)$ and $\sum_{\alpha \in I} a_\alpha \cdot \left(\overline{x}^\alpha - \sum_{j=0}^{m} \lambda(\alpha)_j \cdot e_j \right) \in \mathbf{I}(V)$, we also have

$$\sum_{j=0}^{m} \sum_{\alpha \in I} a_\alpha \cdot \lambda(\alpha)_j \cdot e_j \in \mathbf{I}(V).$$

Therefore, $\sum_{j=0}^{m} \sum_{\alpha \in I} a_\alpha \cdot \lambda(\alpha)_j \cdot [e_j] = 0$. Since $\{[e_0], \ldots, [e_m]\}$ is a basis, we conclude that $\sum_{\alpha \in I} a\lambda(\alpha)_j = 0$ for each $j \in \{0, \ldots, m\}$. Hence,

$$P = \sum_{\alpha \in I} a_\alpha \cdot \left(\overline{x}^\alpha - \sum_{j=0}^{m} \lambda(\alpha)_j \cdot e_j \right) \in \langle \mathbf{I}(V) \cap k_0[x_1, \ldots, x_n] \rangle.$$

Now, it is clear that every $f \in \operatorname{Aut}(K/k_0)$ leaves V invariant. We prove that, if $f \in \operatorname{Aut}(K)$ leaves V invariant, f fixes k_0. Let $\widetilde{f} : R \to R$ given by $\widetilde{f}([\sum a_\alpha \overline{x}^\alpha]) = [\sum f(a_\alpha)\overline{x}^\alpha]$. Then, $\widetilde{f}$ is well defined since f leaves V invariant. Since $f(1) = 1$, we conclude that $\widetilde{f}([\overline{x}^r]) = [\overline{x}^r]$ and that $\widetilde{f}([e_i]) = [e_i]$. Since $\{[e_0], \ldots, [e_m]\}$ is a basis, we conclude that $f(\lambda) = \lambda$ for every $\lambda \in k_0$. □

Theorem 4.10. *Let $V \subseteq K^n$ be a Zariski closed set and $k \subseteq K$ a subfield. Then, there is a formula $\varphi \in \operatorname{For} L_r(k)$ such that $V = \varphi(K)$ if and only if V/k_{perf}.*

Proof. If V/k_{perf}, then there is $\varphi \in \operatorname{For} L_r(k)$ such that $\varphi(K) = V$. Indeed, there are $P_1, \ldots, P_m \in k_{\text{perf}}[x_1, \ldots, x_n]$ such that $V = \mathbf{V}(P_1, \ldots, P_m)$, so V is defined by $P_1(\overline{x}) = 0 \wedge \cdots \wedge P_n(\overline{x}) = 0$. Since the elements of k_{perf} are k-definable, each P_i is k-definable, so V is k-definable.

On the other hand, let K' be a saturated elementary extension of K and $V' = \underline{V}[K']$. Let $k_0 \subseteq K'$ be the subfield given by the lemma 4.9. If there is $\varphi \in \operatorname{For} L_r(k)$ such that $V' = \varphi(K')$, then every automorphism f' fixing k leaves V' invariant. So, f' fixes k_0 by the lemma 4.9. Therefore, $k_0 \subseteq \operatorname{dcl}(k)$ by theorem 1.29. Thus, by lemma 4.8, $k_0 \subseteq k_{\text{perf}}$. So, V'/k_{perf}, hence, V/k_{perf}. □

Now, we want to prove that the Morley's rank is the Krull dimension [Theorem 4.13]. To do that we need the following two lemmas:

Lemma 4.11. *Let $p \in \mathbf{S}_n^K(A)$. Then, there is an A-definable Zariski closed set X such that p is the unique generic type in X over A.*

Proof. Assume that K is saturated and $|K| \geq |A|^+$ and let $\overline{a}$ realize p. Let k be the subfield generated by A. Since $k \subseteq \operatorname{dcl}(A) \subseteq \operatorname{acl}(A)$, $\operatorname{MR}(\overline{a}/A) = \operatorname{MR}(\overline{a}/k)$ [Corollary 2.34]. Let $m = \operatorname{MR}(p)$

By theorem 2.51, $\dim(a_1, \ldots, a_n/k) = m$. Let $\{b_{r_1}, \ldots, b_{r_l}\} \subseteq \{a_1, \ldots, a_n\}$ be such that $\dim(b_{r_1}), \ldots, b_{r_l}/k) = \dim(b_{r_1}, \ldots, b_{r_s}/k) = s$ for $s \leq l \leq n$. We prove that there is $\Delta_{\bar{r}} \subseteq k[x_{r_1}, \ldots, x_{r_l}]$ such that, for any $\bar{c} \in \mathbf{V}(\Delta_{\bar{r}})$ satisfying $\dim(c_{r_1}, \ldots, c_{r_s}/k) = m$, we have $\mathrm{tp}(\bar{c}/k) = \mathrm{tp}(\bar{b}/k)$.

To simplify notation, assume that $\bar{r} = (1, \ldots, l)$. We know that $b_{s+1}, \ldots, b_l \in k(b_1, \ldots, b_s)$. Let $Q_j(b_1, \ldots, b_{j-1}, x_j) \in k(b_1, \ldots, b_{j-1})[x_j]$ be a minimal irreducible polynomial of b_j, for each $s < j \leq l$. Thus, $\mathrm{tp}(b_j/k(b_1, \ldots, b_{j-1}))$ is isolated by Q_j. The coefficients of each $Q_j(b_1, \ldots, b_{j-1}, x_j)$ are k-rational functions of b_1, …, b_{j-1}, so multiplying by the denominators, we may assume that $Q_j(x_1, \ldots, x_{j-1}, x_j)$ is a polynomial. Therefore,

$$\varphi(\overline{x}) = Q_{s+1}(\overline{x}) = 0 \wedge \cdots \wedge Q_l(\overline{x}) = 0$$

is such that $\varphi(b_1, \ldots, b_s, x_{s+1}, \ldots, x_l)$ isolates $\mathrm{tp}(b_{s+1}, \ldots, b_l/k(b_1, \ldots, b_s))$. Then, $\Delta = \{Q_{s+1}, \ldots, Q_l\}$ satisfies our claim. Indeed, let $\bar{c} \in K^l$ such that $K \models \varphi[\bar{c}]$ and $\dim(c_1, \ldots, c_s/k) = s$. By lemma 2.50, $\mathrm{tp}(c_1, \ldots, c_s/k) = \mathrm{tp}(b_1, \ldots, b_s/k)$. So, we may assume that $b_1 = c_1, \ldots, b_s = c_s$ by mapping it via an automorphism which fixes k [lemma 1.28]. Then,

$$Q_{m+1}(b_1, \ldots, b_s, x_{s+1}) \in \mathrm{tp}(c_{s+1}/k(b_1, \ldots, b_s)),$$

so $\mathrm{tp}(c_{s+1}/k(b_1, \ldots, b_s)) = \mathrm{tp}(b_{s+1}/k(b_1, \ldots, b_s))$ because Q_{s+1} isolates this type. Thus, we may assume that $b_{s+1} = c_{s+1}$ by mapping it via an automorphisms which fixes $k(b_1, \ldots, b_s)$. Iterating, we conclude that $\bar{b} = \bar{c}$ via an automorphism which fixes k. So $\mathrm{tp}(\bar{b}/k) = \mathrm{tp}(\bar{c}/k)$.

Now, let

$$\Delta = \bigcup \left\{ \Delta_{\bar{r}} \; : \; \begin{array}{l} \bar{r} \subseteq \{1, \ldots, n\} \text{ such that } \Delta_{\bar{r}} \text{ satisfies the above claim for} \\ \dim(a_{r_1}, \ldots, a_{r_l}/k) = \dim(a_{r_1}, \ldots, a_{r_s}/k) = s \end{array} \right\}.$$

Let $X = \mathbf{V}(\Delta)$. Therefore, any generic element of X over k has the same type that $\overline{a}$ over k. Indeed, let $\bar{b} \in X$ such that $\mathrm{MR}(\bar{b}/k) \geq m$. There is $\bar{r}$ such that $\dim(b_{r_1}, \ldots, b_{r_m}/k) = m$. Then, $\dim(a_{r_1}, \ldots, a_{r_m}/k) = m$ too. Indeed, if a_{r_j} is algebraic respect to $a_{r_1}, \ldots, a_{r_{j-1}}$, then there is a polynomial Q_{r_j} in $\Delta_{r_1, \ldots, r_j}$ such that $Q_{r_m}(a_{r_1}, \ldots, a_{r_{j-1}}, a_{r_j}) = 0$. So b_{r_j} is algebraic respect to $b_{r_1}, \ldots, b_{r_{j-1}}$. Now, $\dim(\bar{b}/k) = \dim(b_{r_1}, \ldots, b_{r_m}/k) = m$ and $\bar{b}$ satisfies $Q(\bar{b}) = 0$ for each $Q \in \Delta_{r_1, \ldots, r_m, r_{m+1}, \ldots, r_n}$, where $\{r_1, \ldots, r_m, \ldots, r_n\} = \{1, \ldots, n\}$. So, $\mathrm{tp}(\bar{b}/k) = \mathrm{tp}(\overline{a}/k)$.

Finally, since $k \subseteq \mathrm{dcl}(A)$, X is A-definable. Since $k \subseteq \mathrm{acl}(A)$, corollary 2.35 implies that the generic elements in X over A are generic over k. So, if $\bar{b}$ is generic in X over A, then $\mathrm{tp}(\bar{b}/k) = \mathrm{tp}(\overline{a}/k)$ and, in particular, $\mathrm{tp}(\bar{b}/A) = \mathrm{tp}(\overline{a}/A)$. □

Lemma 4.12. *Let $k \subseteq K$ a subfield and V/k an affine variety, i.e., an irreducible Zariski closed set. Then,* $\mathrm{Md}(V) = 1$ *and the unique generic type of V over k_{perf} is aximatized by*

$$\{\underline{V}\} \cup \{\neg \underline{W} \; : \; W \subset V \text{ such that } W/k \text{ is a Zariski closed set}\}.$$

Proof. Let p be the type with parameters k_{perf} containing

$$\{\underline{V}\} \cup \{\neg \underline{W} \,:\, W \subset V \text{ such that } W/k \text{ is a Zariski closed sets}\}.$$

By the lemma 4.11, there is a k_{perf}-definable Zariski closed set X such that p is the unique generic type of X over k_{perf}. Let $\mathrm{MR}(p) = m = \mathrm{MR}(X)$. Since $\underline{X} \in p$ is closed and X/k [theorem 4.10], $V \subseteq X$. So $\mathrm{MR}(p) \leq \mathrm{MR}(V) \leq \mathrm{MR}(X)$, i.e., $\mathrm{MR}(V) = m$. Thus, p is the unique generic type in V over k_{perf}. Therefore, V has a unique generic type, i.e., $\mathrm{Md}(V) = 1$. $\square$

Theorem 4.13. *Let V be an affine variety. Then,* $\mathrm{MR}(V) \geq n+1$ *if and only if there is $W \subset V$ subvariety such that* $\mathrm{MR}(W) \geq n$. *Moreover,* $\mathrm{MR}(V) = \dim_{\mathrm{Kr}}(V)$.

Proof. ($\Leftarrow$) Let $W \subset V$ be such that $\mathrm{MR}(W) \geq n$. Of course, $\mathrm{MR}(V) \geq n$. Let k be such that V/k and W/k. Suppose that $\mathrm{MR}(V) = n = \mathrm{MR}(W)$. Let p, q be the generic types in W and V over k_{perf}, respectively. Then, $p = q$ since both are generic types in V over k_{perf} and there is just one by the lemma 4.12. So $\underline{W} \in q$, i.e., $V \subseteq W$ by lemma 4.12, a contradiction. Then, $\mathrm{MR}(V) \geq n+1$.

($\Rightarrow$) Assume $\mathrm{MR}(V) = n+1$. Let $k \subseteq K$ be a subfield such that V is k-definable. Let K' be a $|k|^+$-saturated elementary extension of K and $V' = \underline{V}[K']$. Let $\overline{a} \in V'$ be generic over k. Thus, $\mathrm{MR}(\overline{a}/k) = n+1$. By theorem 2.51, $\dim(\overline{a}/k) = n+1$. Assume that $a_1 \notin \mathrm{acl}(k)$, then $q = \mathrm{tp}(a_1, \ldots, a_n/\mathrm{acl}(k(a_1)))$ is a stationary type such that $\mathrm{MR}(q) = n$. By the lemma 4.11, there is a $\mathrm{acl}(k(a_1))$-definable Zariski closed set W such that q is the unique generic type in W over $\mathrm{acl}(k(a_1))$. Then, $\mathrm{MR}(W) = n$. We may assume that $W \subset V$ since $\underline{V} \in q$. On the other hand, we may assume that W is irreducible. Indeed, there is a partition $W = V_1 \cup V_2 \cup \cdots \cup V_d$ in irreducible sets and $\mathrm{MR}(W) = \max\{\mathrm{MR}(V_i) \,:\, i \leq d\}$. So we may assume that W is just one V_i.

Therefore, $\mathrm{MR}(V) = \dim_{\mathrm{Kr}}(V)$. Indeed, $\mathrm{MR}(V) \geq n$ if and only if there is a sequence of varieties $W_0 \subset W_1 \subset W_2 \subset \cdots \subset W_n = V$ such that $\mathrm{MR}(W_0) = 0$, $\mathrm{MR}(W_1) = 1$, etc. $\square$

Corollary 4.14. *Let V be an affine variety and $U \subseteq V$ be a relative Zariski open subset. Then,* $\mathrm{MR}(U) = \mathrm{MR}(V)$.

Proof. Since $\mathrm{MR}(V) = \max\{\mathrm{MR}(U), \mathrm{MR}(V \setminus U)\}$, it suffices to prove that $\mathrm{MR}(V \setminus U) < \mathrm{MR}(V)$. Now, the latter is a straightforward consequence of the theorem 4.13, since $\mathrm{MR}(V \setminus U) = n$ implies that $\mathrm{MR}(V) > n$.

4.2 Abstract varieties

An *abstract n-variety* of K is a pair $X := (X, \mathcal{A})$ such that $\mathcal{A} = \{(X_i, f_i)\}_{i \leq m}$ where $\{X_i\}_{i \in m}$ is a cover of X and $f_i : X_i \to V_i$ is a bijective function from X_i to an affine variety $V_i \subseteq K^n$, $f_i(X_i \cap X_j) \subseteq V_i$ is a Zariski relative open set and any *transition map* $f_j \circ f_i^{-1} : f_i(X_i \cap X_j) \to f_j(X_i \cap X_j)$ is an isomorphism for any $i, j \leq m$. A *chart* of $x \in X$ is any $(X_i, f_i) \in \mathcal{A}$ such that $x \in X_i$. Let

$k \subseteq K$ be a subfield. We say that $(X, \{(X_i, f_i)\}_{i\leq m})$ is *defined over* k (X/k) if the variety $f_i(X_i)$ is defined over k and $f_j \circ f_i^{-1} : f_i(X_i \cap X_j) \to f_j(X_i \cap X_j)$ is defined over k for each $i, j \leq m$.

The abstract variety X inherit an initial noetherian topology from the Zariski topology of K^n via the charts. Then, in this topology, $X_1, \dots, X_m$ is an open covering and every chart f_i is a topological homeomorphisms between X_i and $V_i = f_i(X_i)$. Indeed, since the transition maps are isomorphisms, the basic open sets of X in X_i are $f_i^{-1}(W) \subseteq X_i$ such that W is a Zariski relative open set in $V_i := f_i(X_i)$. Then, if $H \subseteq X_i$ is an open set, $H = \bigcup_{n\in I} f_i^{-1}(W_n) = f_i^{-1}(W)$.

Finally, since the topology of an abstract variety is a noetherian topology, we can talk about irreducible sets of abstract varieties and decompositions in irreducible sets.

Remark. Every abstract variety is definable with imaginaries, so is definable in K^n by elimination of imaginaries [Theorem 4.6]. Indeed, consider the disjoint union $V_1 \sqcup \cdots \sqcup V_m$ and the relations $\sim_{ij}$ between the elements of $f_i(X_i \cap X_j)$ and $f_j(X_i \cap X_j)$ via $f_j \circ f_i^{-1}$. Let $a \sim b \Leftrightarrow$ There are ij $a \sim_{ij} b$. Therefore, our variety X is the quotient ${}^{\bigsqcup_{i=1}^m V_i}/_{\sim}$, which is definable with imaginaries. Then, $X_i = {}^{V_i}/_{\sim}$ and $f_i^{-1} : V_i \to X_i$ is the canonical projection. Also, X/k if and only if X is k_{perf}-definable by lemma 4.10. Finally, each chart is a definable bijection, so it preserves Morley's ranks and degrees. Therefore, if X is an irreducible abstract variety, then $\mathrm{MR}(X)$ is the dimension of X, $\mathrm{Md}(X) = 1$ and, for any open subset $U \subseteq X$, $\mathrm{MR}(U) = \mathrm{MR}(X)$.

Example. Any relative Zariski open subset U of a variety $V \subseteq K^n$ is an abstract variety. Indeed, if $U = \{\overline{x} \in V : P_1(\overline{x}) \neq 0 \vee \cdots \vee P_m(\overline{x}) \neq 0\}$, consider the charts (U_i, f_i) where $U_i = \{\overline{x} \in V : P_i(\overline{x}) \neq 0\}$ and $f_i : \overline{x} \mapsto (\overline{x}, P_i(\overline{x})^{-1})$. Then, $(U, \{(U_i, f_i)\}_{i\leq m})$ is an abstract $n+1$-variety. Indeed, $f_i(U_i) = \{(\overline{x}, y) \in K^{n+1} : y \cdot P_i(\overline{x}) - 1 = 0 \wedge \overline{x} \in V\}$ is an affine variety and $f_i \circ f_j^{-1} : (\overline{x}, y) \mapsto (\overline{x}, P_i(\overline{x})^{-1})$ is a morphism where $f_i(U_i \cap U_j) = \{(\overline{x}, y) \in K^{n+1} : y \cdot P_i(\overline{x}) - 1 = 0 \wedge \overline{x} \in V \wedge \overline{x} \in U_j\}$ is a Zariski relative open subset.

Let X and Y be abstract varieties over K. A *morphism* $\Psi : X \to Y$ is continuous function such that, for every $x \in X$ and any charts (X_i, f_i) and (Y_j, g_j) of x and $\Psi(x)$, the function $g_j \circ \Psi \circ f_i^{-1} : f_i(X_i \cap \Psi^{-1}(Y_j)) \to g_j(Y_j \cap \Psi(X_i))$ is an affine morphism. An *isomorphism* is a bijective morphism whose inverse is also a morphism.

Remark. The morphism are definable functions with imaginaries, so are definable by elimination of imaginaries [Theorem 4.6].

Example. The most significant example is the projective space $\mathbb{P}^n(K)$. Indeed, the homogeneous maps,

$$\begin{array}{cccc} \phi_i : & A_i^n & \to & K^n \\ & [x_0, \dots, x_n] & \mapsto & ({}^{x_0}/_{x_i}, \dots, {}^{x_{i-1}}/_{x_i}, {}^{x_{i+1}}/_{x_i}, \dots, {}^{x_n}/_{x_i}) \end{array}$$

where $A_i^n = \{[x_0, \dots, x_n] : x_i \neq 0\}$, are charts of the projective space. Indeed, $\phi_i(A_i^n) = K^n$, $\phi_i(A_i^n \cap A_j^n) = \{\overline{x} \in K^n : x_j \neq 0\}$ and $\phi_j \circ \phi_i^{-1}$ is a morphism.

Of course, any projective variety has an abstract variety structure by restricting this one.

Let X be an abstract n-variety over K. We say that X is *separated* if the diagonal of $X \times X$ is a closed set, and that X is *complete* if, for every variety Y over K, the projection $\pi : X \times Y \to Y$ is a closed map.

Lemma 4.15. ***(Regular functions from complete varieties)*** *Let X be an irreducible complete abstract n-variety over K. Then, every morphism $f : X \to K$ is constant.*

Proof. If $f = 0$, there is nothing to prove. Let $f : X \to K$ be a morphism $f \neq 0$. Let

$$Z = \{(x, y) \in X \times K \;:\; f(x) \cdot y = 1\}.$$

Then, Z is a non-empty closed subset of $X \times K$. Since X is complete, $\pi(Z)$ is a non-empty Zariski closed set of K, so a definable set. Since K is strongly minimal, $\pi(Z)$ is finite or cofinite. Since $\pi(Z)$ is closed, or $\pi(Z) = K$ or $\pi(Z)$ is finite. We know that $0 \notin \pi(Z)$, so $\pi(Z)$ is finite. Suppose that $\pi(Z) = \{\lambda_1, \ldots, \lambda_l\}$, then $X = \bigcup_{j=1}^{l} f^{-1}(\{\lambda_j^{-1}\})$. Since f is a continuous map, each $f^{-1}(\{\lambda_j^{-1}\})$ is a closed set. Since $\mathbf{X}$ is irreducible, $X = f^{-1}(\{\lambda_j^{-1}\})$ for one j. Therefore, $\pi(Z) = \{\lambda\}$ and $f(x) = \lambda^{-1}$ for every $x \in X$. □

Theorem 4.16. ***(Rigidity theorem)*** *Let X, Y and Z be irreducible abstract varieties over K such that X is a complete variety, and let $\Psi : X \times Y \to Z$ be a morphism. If there is an $y_0 \in Y$ satisfying that $\Psi(x, y_0) = \Psi(x', y_0)$, for every $x, x' \in X$, then $\Psi(x, y) = \Psi(x', y)$ for every $x, x' \in X$ and $y \in Y$.*

Proof. Let $z_0 = \Psi(x, y_0)$ and (Z_i, h_i) be chart of z_0. Since Ψ is a continuous map, $\Psi^{-1}(Z_i^c)$ is a closed subset. Since X is complete, $\pi(\Psi^{-1}(Z_i^c))$ is a closed set. Note that $y_0 \notin \pi(\Psi^{-1}(Z_i^c))$, so $U = Y \setminus \pi(\Psi^{-1}(Z_i^c))$ is a non-empty open set. For any $x \in X$ and $y \in U$, we have that $\Psi(x, y) \in Z_i$ and $x \mapsto \Psi(x, y)$ is a morphism. Now, $h_i \circ \Psi(-, y) : X \to K^n$ is constant by the lemma 4.15. So, since $h_i : Z_i \to K^n$ is a one-to-one map, for every $y \in U$ and any $x, x' \in X$, $\Psi(x, y) = \Psi(x', y)$. Let $x, x' \in X$ and $U' = \{y \in Y \;:\; \Psi(x, y) = \Psi(x', y)\}$. It is clear that $U \subseteq U'$ and U' is a closed set. Since Y is irreducible, $U^c \cup U' = Y$ implies that $U^c = Y$ or $U' = Y$. Since we know that $U^c \neq Y$, we obtain that $\{y \in Y \;:\; \Psi(x, y) = \Psi(x', y)\} = U' = Y$. □

Let X be an abstract variety and $x \in X$. We say that x is a *singular point* if, for every chart (X_i, f_i) of x, the point $f_i(x)$ is singular in $f_i(X_i)$. Since affine morphisms take singular points to singular points, it suffices to check one chart to verify whether a point is singular.

Remark. The definition of abstract variety is analogous to the definition of topological manifold. Moreover, the abstract varieties over $\mathbb{C}$, without singular points, are also differential manifolds.

4.3 Algebraic groups and abelian varieties

An *algebraic group* is a group $(G, \cdot, ^{-1})$ where G is an abstract variety and $\cdot : G \times G \to G$ and $-^{-1} : G \to G$ are a morphism.

Notation. In the rest of this section and except otherwise stated, $(G, \cdot)$ will denote an algebraic group.

Lemma* 4.17.** ***(Separability and smoothly) *Every algebraic group is separable and does not have singular points.*

Proof. Let $\Psi : G \times G \to G$ be defined as $\Psi(x, y) = x \cdot y^{-1}$. Then, Ψ is a morphism and the diagonal of $G \times G$ is $\Psi^{-1}(\{e\})$. Therefore, by continuity of Ψ, we conclude that G is separable. On the other hand, we know that the set of singular points is a proper closed subset, so there is a non-singular point $a \in G$. Hence, since $a \cdot b^{-1} \cdot - : G \to G$ is a morphism for every $b \in G$, we conclude that b is a non-singular point for every $b \in G$. Indeed, if b is singular, $a \cdot b^{-1} \cdot b = a$ is singular too, a contradiction. $\square$

Example. The standard example of algebraic group is the linear group of matrices $\mathrm{GL}_n(K)$, which is a Zariski open subset of K^{n^2} defined by the inequality $\det(A) \neq 0$. Of course, the product of matrices is a morphism. Since $\mathrm{GL}_n(K)$ is also isomorphic to an irreducible affine variety of K^{n^2+1}, we may consider $\mathrm{GL}_n(K)$ as an algebraic group. Indeed, $\mathrm{GL}_n(K) = \{\overline{x} \in K^{n^2} : \det(\overline{x}) \neq 0\}$ and $\Psi : \overline{x} \mapsto (\overline{x}, \det(\overline{x})^{-1})$ is an isomorphism between $\mathrm{GL}_n(K)$ and $\{(\overline{x}, y) \in K^{n^2+1} : \det(\overline{x}) \cdot y = 1\}$.

Algebraic groups are definable groups in algebraically closed fields. On the other hand, every definable group is definable isomorphic to an algebraic group (see Proposition 4.12 of [4]).

Proposition 4.18. *Let $H \leq G$ be a definable subgroup. Then, H is closed.*

Proof. We know that $\overline{H} = \bigcup_{i \in I} Y_i$ where I is finite, the Y_i's are pairwise disjoint and Y_i are irreducible. Thus,

$$H = \bigcup_{i \in I} Y_i \setminus Z_i$$

where Z_i is a proper closed subset of Y_i. Then, $\mathrm{Md}(Y_i) = 1$ and $\mathrm{MR}(Z_i) < \mathrm{MR}(Y_i)$ for each $i \in I$. We know that $\overline{H}$ is a definable group. Let $\alpha = \max\{\mathrm{MR}(Y_i) : i \in I\}$, let $I_0 \subseteq I$ be such that $\mathrm{MR}(Y_i) = \alpha$ for every $i \in I_0$ and let $U = \bigcup_{i \in I_0} Y_i \setminus Z_i$. Then,

$$\overline{H} \setminus U \subseteq \bigcup_{i \in I \setminus I_0} Y_i \cup \bigcup_{i \in I_0} Z_i.$$

Thus, $\mathrm{MR}(\overline{H} \setminus U) < \alpha = \mathrm{MR}(\overline{H})$. Therefore, $\mathrm{MR}(U) = \mathrm{MR}(\overline{H})$ and $\mathrm{Md}(U) = \mathrm{Md}(\overline{H})$, so $U \cdot U = \overline{H}$ by proposition 3.13. Hence, $H = \overline{H}$ since $U \subseteq H$. $\square$

Proposition 4.19. *G° is the irreducible component of G containing the identity. Thus, G is connected in model-theoretic sense if and only if G is a variety, and if and only if G is Zariski connected.*

Proof. It is a basic algebraic-geometric fact that the maximal irreducible component of G containing the identity is a connected group of finite index. Therefore, it is G°. □

Now, thanks to the theory studied about groups with Morley's rank, we know that, for example,

- the centralizer of a set in an algebraic group is an algebraic group and is the centralizer of a finite number of points [example 3.1]; or
- the commutator of an algebraic group is an algebraic group and if the group is irreducible the commutator is it too [corollary 3.17].

An *abelian variety* $(A, +)$ is a irreducible (connected) complete algebraic group.

Example. Elliptic curves are abelian varieties. Indeed, it is clear that elliptic curves are irreducible algebraic groups. On the other hand, it is a well known fact that projective varieties are complete.

***Lemma* 4.20.** *Every abelian variety is an abelian group.*

Proof. Let $f : A \times A \to A$ be given by $f(a, b) = a + b - a$. We have that $f(a, b) = f(a', b)$ for every $a, a'b \in A$ by the rigidity theorem [Theorem 4.16], since A is a complete and separable variety [Lemma 4.17] and $f(a, 0) = 0$ for every $a \in A$. Therefore, for any $a, b \in A$, we have that $f(a, b) = f(b, b) = b$, i.e., A is an abelian group. □

***Lemma* 4.21.** *Let $(G, \cdot)$ be an abelian variety, $(H, *)$ a commutative algebraic group and $f : G \to H$ a morphism. Then, $h : G \to H$ given by $h(x) = f(x) - f(e)$ is an homomorphism of groups.*

Proof. Consider $g : G \times G \to H$ given by $g(x, y) = f(x) + f(y) - f(x \cdot y)$, which is a morphism. Then, $g(e, y) = f(e) + f(y) - f(y) = f(e) = f(x) + f(e) - f(x) = g(x, e)$, for any $x, y \in G$. By the rigidity theorem [Theorem 4.16], we have that $g(x, y) = g(x', y') = f(e)$ for any $x, y, x', y' \in G$. Therefore,

$$\begin{aligned} h(x \cdot y) =& f(x \cdot y) - f(e) = -g(x, y) + f(x) + f(y) - f(e) = \\ =& - f(e) + f(x) + f(y) - f(e) = \\ =& (f(x) - f(e)) + (f(y) - f(e)) = h(x) + h(y). \end{aligned}$$

□

Let V be a $\mathbb{C}$-vector space of finite dimension d. A *complex torus* is a quotient group ${}^{V}/_{\Lambda}$ where $\Lambda \subseteq V$ is a lattice of rang $2d$, i.e., a discrete subgroup aditivo.

Remark. Complex torus are abelian varieties.

Theorem 4.22. ***(Complex abelian varieties)*** *Every complex abelian variety is a torus.*

Proof. It suffices to note that abelian varieties over $\mathbb{C}$ are Lie groups since abelian varieties do not have singular points [Lemma 4.17]. Therefore, every abelian variety over $\mathbb{C}$ is a torus, since Lie groups are torus. $\square$

5 The Mordell-Lang's Conjecture

5.1 Introduction

Mordell's Conjecture (1922): If C is an smooth projective curve in $\mathbb{C}$ defined over $\mathbb{Q}$ of genus greater or equal that 2, then $C(\mathbb{Q})$ is finite.

Therefore, Mordell's Conjecture is a problem about diophantine equations. Given a polynomial $P \in \mathbb{Q}[x,y]$, define the curve $X = \{(x,y) \in \mathbb{C}^2 : P(x,y) = 0\}$. All in all, we want to determinate $X(\mathbb{Q})$. An approach is to consider the projective smooth curve C associated to the homogenization of P and to study $C(\mathbb{Q})$. Therefore, there are three cases:

1. If C has genus 0, C is a *rational curve*. Then, either $C(\mathbb{Q}) = \emptyset$ (e.g., $P(x,y) = x^2 + y^2 + 1$) or all but finitely many solutions are parametrized by rational functions $x(t)$ and $y(t)$ (e.g., $P(x,y) = x^2 + y^2 - 1$ and $x(t) = {}^{2t}/_{t^2+1}$ and $y(t) = {}^{t^2-1}/_{t^2+1}$).

2. If C has genus 1, then either $C(\mathbb{Q}) = \emptyset$ or C is an elliptic curve taking a point as origin. Therefore, we have that $C(\mathbb{Q})$ is a finitely generated group by the Mordell-Weil's theorem.

3. If C has genus greater or equal 2, we apply the Mordell's Conjecture: $C(\mathbb{Q})$ is finite.

Hence, the Mordell's Conjecture is a complete classification of the diophantic problem of determining the rational solutions of projective curves.

Absolute Mordell-Lang's Conjecture (characteristic 0): Let X be an irreducible subvariety of a complex abelian variety A and Γ a subgroup of finite rank, i.e., there is a finitely generated subgroup $\Gamma_0 \leq \Gamma$ such that, for every element $\gamma \in \Gamma$, there is $n \in \mathbb{N}$ satisfying $n\gamma \in \Gamma_0$ and $n1 \neq 0$. Then, there are $\gamma_1, \ldots, \gamma_m \in \Gamma$ and $B_1, \ldots, B_m$ abelian subvarieties of A such that $\gamma_i + B_i \subseteq X$, for each $i \in \{1, \ldots, m\}$, and

$$\Gamma \cap X = \bigcup_{i=1}^{m} \gamma_i + B_i \cap \Gamma.$$

Note that this is actually a diophantic conjecture. Indeed, if K is a field finitely generated over $\mathbb{Q}$ and $X \subseteq A$ is an irreducible subvariety over K of an abelian variety A over K, then $A(K)$ is finitely generated by the Mordell-Weil's theorem. Thus, taking $\Gamma = A(K)$, the absolute Mordell-Lang's conjecture describes the K-points of X.

The Mordell-Lang conjecture in absolute form is equvialent to the following.

Lemma 5.1. ***(First equivalence)*** *The absolute Mordell-Lang's conjecture is equivalent to the following statement:*

Let X be an irreducible subvariety of a complex abelian variety A and $\Gamma \leq A$ a subgroup of finite rank. Then, if $X \cap \Gamma$ is Zariski dense in X, X is the translate of an abelian subvariety by a point of Γ.

Lemma 5.2. ***(Second equivalence)*** *The absolute Mordell-Lang's conjecture is equivalent to the following statement:*

Let X be an irreducible subvariety of a complex abelian variety A and $\Gamma \leq A$ a subgroup of finite rank. If $X \cap \Gamma$ is dense in X and $\mathrm{Stab}_X = \{0\}$, then X is a translate of an abelian subvariety by a point of Γ.

Proposition 5.3. *The absolute Mordell-Lang's conjecture implies the Mordell's conjecture.*

Remark. **(Manin-Mumford conjecture)** The Mordell-Lang's conjecture is also a generalization of the Manin-Mumford conjecture which states that either $C \cap A_{\text{torsion}}$ is finite or C is a translate of an elliptic curve, for any curve C in an abelian variety A.

5.2 Model-theoretic content of Mordell-Lang's conjecture

Let $K \models \mathrm{ACF}$, A a commutative algebraic group over K and $\Gamma \leq A$. We say that (K, A, Γ) is of Lang-type if, for any $n \in \omega$ and every subvariety $X \subseteq A^n (= A \times \overset{n}{\cdots} \times A)$ over K, $X \cap \Gamma^n$ is a finite union of cosets.

Thus, the absolute Mordell-Lang's conjecture says that, for any subgroup Γ of finite rank, $(\mathbb{C}, A, \Gamma)$ is of Lang-type.

Lemma 5.4. *Let $K \models \mathrm{ACF}$, G be an algebraic group over K, X a subvariety of G and Γ a subgroup of G. Then, $X \cap \Gamma$ is a finite union of cosets if and only if there are connected algebraic subgroups $G_1, \ldots, G_n$ of G and respectively translates $C_1, \ldots, C_n$ of these ones such that $C_i \subseteq X$, for each i, and $X \cap \Gamma \subseteq C_1 \cup \cdots \cup C_n$.*

Proof. ($\Leftarrow$) If $X \cap \Gamma \subseteq C_1 \cup \cdots \cup C_n \subseteq X$, then $X \cap \Gamma = \bigcup_{i=1}^n C_i \cap \Gamma$. Let $C_i = g_i G_i$ for each i. Then, either $C_i \cap \Gamma = \emptyset$ or we may assume $g_i \in \Gamma$. The latter proves that $X \cap \Gamma$ is a finite union of cosets of algebraic subgroups restricted to Γ^n.

($\Rightarrow$) If $X \cap \Gamma = B_1 \cup \cdots \cup B_n$ with $B_i = g_i H_i$ where $H_i \leq G$ not necessarily closed. Consider $\overline{B_i} = g_i \overline{H_i} \subseteq X$. Then, $\overline{H_i} \leq G$ is a Zariski closed set. The connected component $\overline{H_i}^{\circ}$ is a variety [proposition 4.19]. Therefore, for some $c_i j$,

$$X \cap \Gamma \subseteq \bigcup_i^n \bigcup_j g_i c_{ij} \overline{H_i}^{\circ} \subseteq X.$$

□

Lemma 5.5. ***(Neumann's lemma)*** *Let G be a group, $K_1, \ldots, K_n \leq G$ different subgroups and $a_1^1, \ldots, a_{m_n}^n \in G$ such that $G = \bigcup_{i=1}^n \bigcup_{j=1}^{m_i} a_j^i \cdot K^i$. Then, $G = \bigcup_{i \in \Delta} \bigcup_{j=1}^{m_i} a_{j,i} K_i$ for*

$$\Delta = \{i \in \{1, \ldots, n\} \, : \, [G : K_i] \in \mathbb{N}\}.$$

In particular, $\Delta \neq \emptyset$, i.e., at least one of the factors has finite index.

Proof. Firstly, we prove that $\Delta \neq \emptyset$ by induction of n. For $n = 1$, we have that $G = \bigcup a_j K_1$, so K_1 has finite index. For $n > 1$, since there is nothing to prove if $G = \bigcup_{j=1}^{m_n} a_{j,n} K_n$, let $h \in G$ be such that $h \notin \bigcup_{j=1}^{m_n} a_{j,n} K_n$. Then, $hK_n \subseteq \bigcup_{i=1}^{n-1} \bigcup_{j=1}^{m_i} a_{j,i} K_i$. So,

$$G = \left(\bigcup_{i=1}^{n-1} \bigcup_{j=1}^{m_i} a_{j,i} K_i \right) \cup \bigcup_{j=1}^{m_n} a_{j,n} \cdot h^{-1} \bigcup_{i=1}^{n-1} \bigcup_{s=1}^{m_i} a_{s,i} K_i.$$

By induction hypothesis, at least one of $K_1, \ldots, K_{r-1}$ has finite index.

Now, we prove the lemma. Assume that $\Delta = \{1, \ldots, r\}$ with $1 \leq r \leq n$. Let $H = \bigcap_{i \in \Delta} K_i$. It is clear that H is a subgroup of finite index. Thus, each K_i for $i \in \Delta$ is a finite union of cosets of H. So, there are $b_1, \ldots, b_N \in G$ such that

$$\bigcup_{i=1}^{r} \bigcup_{j=1}^{m_i} a_{j,i} K_i = \bigcup_{j=1}^{N} b_j H.$$

Suppose that $G \neq \bigcup b_j H$ and let $h \in G$ such that $b \notin \bigcup b_j H$. Then,

$$bH \subseteq \bigcup_{i=r+1}^{n} \bigcup_{j=1}^{m_i} a_{j,i} K_i.$$

So, $H = \bigcup \bigcup b^{-1} \cdot a_{j,i} K_i \cap H$. Thence, we have proved that there is $i \notin \Delta$ such that $K_i \cap H$ has finite index in H. Thus, for one $i \notin \Delta$, K_i has finite index in G, a contradiction. $\square$

Lemma 5.6. *Let $K \models$ ACF countable, A be a commutative algebraic group over K and $\Gamma \leq A$. Let $\mathfrak{A} = (\Gamma, X \cap \Gamma^n \ : \ X \subseteq A^n$ K-def in $(K, +, \cdot))$ and $T_0 = \mathrm{Teo}(\mathfrak{A})$. Then, if (K, A, Γ) is of Lang-type, T_0 is a totally transcendental one-based theory.*

Proof. After proving that it is totally transcendental, we are going to apply theorem 3.22. So, we first prove that every definable set in $\mathfrak{A}$ is a boolean combination of cosets.

Firstly, every $X \subseteq A^n$ K-definable in $(K, +, \cdot)$ is a finite union of varieties. Since (K, A, Γ) is of Lang-type, for $X \subseteq A^n$ variety, we know that $X \cap \Gamma^n$ is a finite union of cosets $B_i \cap \Gamma^n$ where $B_i \leq A^n$ is an algebraic subgroup. Hence, every definable set in $\mathfrak{A}$ is already definable in $\mathfrak{A}_0 = (\Gamma, \gamma B \cap \Gamma^n)_{B \leq A^n \ K-\text{def in } (K,+,\cdot) \text{ and } \gamma \in \Gamma^n}$. Then, it suffices to prove that $\mathrm{Teo}(\mathfrak{A}_0)$ has quantifier elimination. Note the following:

i. The intersection of cosets is empty or a coset.

ii. The projection on Γ^n of a coset $\gamma B \cap \Gamma^{n+1}$ is the coset of the projection of $B \cap \Gamma^{n+1}$ in which the projection of γ is. Indeed, he projection of B is an algebraic subgroup of A^n.

iii. The diagonal is a coset, we write $D \cap \Gamma^2$.

iv. Adding a dummy variable to a coset $\gamma B \cap \Gamma^n$ of Γ^n produces a coset of Γ^{n+1}, this is $(\gamma, e)(B \times \{e\}) \cap \Gamma^{n+1}$. Also, changing names of variables in a coset gives a coset (e.g., $\underline{\gamma B \cap \Gamma^n}(\overline{x}, y, y)$ is equivalent to $\exists z(\underline{\gamma B \cap \Gamma^n}(\overline{x}, y, z) \wedge \underline{D}(y, z))$).
v. Let $\gamma B \cap \Gamma^n$, $P(\overline{y}) = \{x : (x, \overline{y}) \in \gamma B \cap \Gamma^n\}$ and $\overline{a}, \overline{b} \in \Gamma^{n-1}$. Then, either $P(\overline{a}) \cap P(\overline{b}) = \emptyset$ or $P(\overline{a}) = P(\overline{b})$. Moreover, if $P(\overline{a}) \neq \emptyset$, then it is a coset of $H = \{x : (x, e, \ldots, e) \in B \cap \Gamma^n\}$. Indeed, let $x_0 \in P(\overline{a})$, it is clear that $x_0 H \subseteq P(\overline{a})$. Let $x_1 \in P(\overline{a})$. Then, $(x_0, \overline{a}), (x_1, \overline{a}) \in \gamma B \cap \Gamma^n$, so $x_1^{-1} x_0 \in H$.

To prove quantifier elimination, it suffices to prove that $\forall x \psi(x, \overline{y})$ is equivalent to a boolean combination of cosets if ψ is so. And, using **iv.** and **i.**, this reduces to the case:

$$\psi = \bigwedge_i \left(\underline{\gamma_i B_i \cap \Gamma^n}(x, \overline{y}) \to \bigvee_j \underline{\gamma_{ij} B_{ij} \cap \Gamma^n}(x, \overline{y}) \right).$$

Also, by **i.**, we may assume that for each i, j

$$\mathfrak{A}_0 \models \forall x \forall \overline{y} \left(\underline{\gamma_{ij} B_{ij} \cap \Gamma^n}(x, \overline{y}) \to \underline{\gamma_i B_i \cap \Gamma^n}(x, \overline{y}) \right).$$

Finally, the quantifier $\forall$ satisfies that $\forall x \psi$ is equivalent to

$$\bigwedge_i \forall x \left(\underline{\gamma_i B_i \cap \Gamma^n}(x, \overline{y}) \to \bigvee_j \underline{\gamma_{ij} B_{ij} \cap \Gamma^n}(x, \overline{y}) \right),$$

so we may assume that there is just one i. Hence, it suffices to prove that there is a boolean combination of cosets equivalent to

$$\forall x \left(\underline{\gamma_i B_i \cap \Gamma^n}(x, \overline{y}) \to \bigvee_j \underline{\gamma_{ij} B_{ij} \cap \Gamma^n}(x, \overline{y}) \right).$$

Let $\overline{a}$ from Γ. Let $P(\overline{a}) = \gamma B \cap \Gamma^n(x, \overline{a})[\mathfrak{A}_0]$ and $N_j(\overline{a}) = \gamma_j B_j \cap \Gamma^n(x, \overline{a})[\mathfrak{A}_0]$, for each j. Hence, it suffices to prove that there is a boolean combination of cosets such that $P(\overline{a}) = \bigcup_j^J N_j(\overline{a})$ if and only if $\overline{a}$ belongs to it. Let $H = B \cap \Gamma^n(x, \overline{e})[\mathfrak{A}_0]$ and $H_j = B_j \cap \Gamma^n(x, \overline{e})[\mathfrak{A}_0]$. By **v.**, note that $P(\overline{a})$ and $N_j(\overline{a})$ are respectively cosets of H and H_j or empty, for any $\overline{a}$ and any j. Let $\Delta = \{j \in \{1, \ldots, J\} : [H : H_j] \in \mathbb{N}\}$. Let $K = \bigcap_{j \in \Delta} H_j$, $\widetilde{A} = {}^{H}/_{K}$ and $\widetilde{A}_j = {}^{H_j}/_{K} := \{xH : x \in H_j\}$ for each $j \in \Delta$. Note that K, $\widetilde{A}$ and $\widetilde{A}_j$ do not depend on $\overline{a}$.

There are two cases, either $P(\overline{a}) = \emptyset$ or $P(\overline{a}) \neq \emptyset$. If $P(\overline{a}) = \emptyset$, then $P(\overline{a}) \subseteq \bigcup_j^J N_j(\overline{a})$, and $P(\overline{a}) = \emptyset$ if and only if $\overline{a}$ is in $\neq \exists x \underline{P}(x, \overline{y})[\mathfrak{A}_0]$, which is a coset by **ii.**. For the other case, let $I(\overline{a}) = \{j : N_j(\overline{a}) \neq \emptyset\}$. Then, $P(\overline{a}) = \bigcup_{j \in I} N_j(\overline{a})$ if and only if $H = \bigcup_{j \in I(\overline{a})} \alpha_j H_j$ for some $\alpha_1, \ldots, \alpha_J$. By the Neumann's lemma 5.5, $H = \bigcup_{j \in \Delta \cap I(\overline{a})} \alpha_j H_j$, so $P(\overline{a}) = \bigcup_{j \in \Delta \cap I(\overline{a})} N_j(\overline{a})$. Let $A = x_0 \widetilde{A} = \{x_0 C : C \in \widetilde{A}\}$ and $A_j = x_j \widetilde{A}_j$ where $x_0 \in P(\overline{a})$ and $x_j \in N_j(\overline{a})$

for $j \in \Delta \cap I(\overline{a})$. Thus, $\emptyset \neq P(\overline{a}) = \bigcup_j N_j(\overline{a})$ if and only if $A = \bigcup_{j\in\Delta\cap I(\overline{a})} A_j$. By the Inclusion-exclusion principle, $A = \bigcup_{j\in\Delta\cap I(\overline{a})} A_j$ if and only if

$$\sum_{T\subseteq\Delta\cap I(\overline{a})} (-1)^{|T|} \left| \bigcap_{j\in T} A_j \right| = 0.$$

Finally, note that $\bigcap_{j\in T} A_j \neq \emptyset$ if and only if $\bigcap_{j\in T} N_j(\overline{a}) \neq \emptyset$. Thus, $|\bigcap_{j\in T} A_j| = |\bigcap_{j\in T} \widetilde{A}_j|$. Therefore, we conclude that $P(\overline{a}) = \bigcup_j N_j(\overline{a})$ if and only if $\overline{a}$ is in the coset $\neq \exists x\underline{P}(x,\overline{y})[\mathfrak{A}_0]$ or

$$\sum_{T\in\Sigma(\overline{a})} (-1)^{|T|} \left| \bigcap_{j\in T} \widetilde{A}_j \right| = 0,$$

where $\Sigma(\overline{a})$ is the set of $T \subseteq \Delta \cap I(\overline{a})$ such that $\bigcap_{j\in T} N_j(\overline{a})$. Now, since the numbers $\left|\bigcap_{j\in T} \widetilde{A}_j\right|$ do not depend on $\overline{a}$, we have that there are specific sets $\Sigma_1, \ldots, \Sigma_N$ for which the equation is true. Hence, $P(\overline{a}) = \bigcup_j N_j(\overline{a})$ if and only if $\overline{a}$ is in the coset $\neq \exists x\underline{P}(x,\overline{y})[\mathfrak{A}_0]$ or $\Sigma(\overline{a}) \in \{\Sigma_1, \ldots, \Sigma_n\}$, and $\Sigma(\overline{a}) = \Sigma_i$ is actually that $\overline{a}$ is in a specific boolean combination of the sets $\exists x \bigcap_{j\in T} N_j(x,\overline{y})[\mathfrak{A}_0]$, which are cosets by **ii.**.

Now, we prove that T_0 is totally transcendental to apply 3.22. It suffices to prove that it is ω-stable [theorem 2.18]. Given a type $p \in \mathbf{S}_n^{\mathfrak{C}^\Gamma}(C)$ with $\mathrm{card}(C) \leq \omega$, consider

$$\Sigma_p = \{\underline{\mathbf{B}}, \neg\underline{\mathbf{B}'} : \mathbf{B}, \mathbf{B}' \leq \mathbf{A}^n \ K \cup C\text{-def. in } \mathfrak{C}^K \text{ and } \underline{\mathbf{B} \cap \Gamma^n}, \neg\underline{\mathbf{B} \cap \Gamma^n} \in p\}.$$

By quantifier elimination, for any $p, q \in \mathbf{S}_n^{\mathfrak{C}^\Gamma}(C)$, $p = q \Leftrightarrow \Sigma_p = \Sigma_q$. Also, note that p finitely satisfiable in $\mathfrak{C}^\Gamma$ implies Σ_p finitely satisfiable in $\mathfrak{C}^K$. Since $\mathfrak{C}^K$ is ω-stable and $\mathrm{card}(K) = \aleph_0$, we conclude that T_0 is ω-stable too. Finally, applying 3.22, T_0 is one-based. □

Remark. Actually, without the condition $\mathrm{card}(K) = \aleph_0$, the lemma is still true replacing totally transcendence by stable. Indeed, most of the theory studied in the sections 3 and 4 of the chapter 2 is easy adapted in the context of stable theories not necessarily totally transcendent. In this context, forking is defined by heirs and coheirs [theorem 2.42] and preserves its fundamental properties [proposition 2.31, theorem 2.33]. Also, the significant result 2.29, which gives us canonical bases, is also true in general stable theories, so the result 2.37 and some variations of its corollaries are true too. Therefore, the theorem 3.22 is also true for stable theories and the last proof can be adapted.

Theorem 5.7. *Let $K \models$ ACF countable, A a commutative algebraic group over K and $\Gamma \leq A$. Then, (K, A, Γ) is of Lang-type if and only if the theory $T = \mathrm{Teo}(K, +, \cdot, \Gamma, a)_{a\in K}$ is totally transcendental and the formula $\underline{\Gamma}$ is one based in T.*

Proof. ($\Leftarrow$) Let $\mathfrak{A} = (\Gamma, X \cap \Gamma^n \ : \ X \subseteq A^n \ K\text{-def in } (K,+,\cdot))$ and $\mathfrak{B}_{\Gamma,K} = (K,+,\cdot,\Gamma,a)_{a\in K}$. Let $X \subseteq A$ be a subvariety. Since $X \cap \Gamma^n$ is a definable set in the structure $\mathfrak{B}$, by the theorem 3.22, $X \cap \Gamma^n$ is a finite boolean combination of definable cosets of A^n. Thus, $\overline{X \cap \Gamma^n}$ is a finite union of cosets of algebraic subgroups. By the lemma 5.4, we conclude that $X \cap \Gamma^n$ is a finite union of cosets of algebraic subgroups restricted to Γ. Therefore, (K, A, Γ) is of Lang-type.

($\Rightarrow$) Let $\mathfrak{C}^K_{\Gamma,K} = (\mathbf{K},+,\cdot,\mathbf{\Gamma},a)_{a\in K}$ be the monster extension of $\mathfrak{B}_{\Gamma,K}$ and $\mathfrak{C}^\Gamma = (\mathbf{\Gamma}, \mathbf{X} \cap \mathbf{\Gamma}^n \ : \ X \subseteq A \ K\text{-def. in } (K,+,\cdot))$ of $\mathfrak{A}$. Note that $\mathfrak{C}^\Gamma$ is totally transcendental and one-based by lemma 5.6. Let acl_f denote the algebraic closure in the field-theoretic sense:

Claim: Let $B \subseteq \mathbf{K}$ be an infinite subset. Then, there is a subset $C \subseteq \mathbf{\Gamma}$ such that $\mathrm{card}(C) = \mathrm{card}(B)$ and, for any $\overline{a}_1$ and $\overline{a}_2$ from $\mathbf{\Gamma}$ with the same type over C in $\mathfrak{C}^\Gamma$, there is a map $\mathbf{f}: \ \mathrm{acl}_f(B \cup K \cup \mathbf{\Gamma}) \to \mathrm{acl}_f(B \cup K \cup \mathbf{\Gamma})$ satisfying the following properties:

i. $\mathbf{f}$ fixes $B \cup K \cup C$ pointwise.

ii. $\mathbf{f}_{|\mathbf{\Gamma}}$ is an automorphism of the structure $\mathfrak{C}^\Gamma$.

iii. $\mathbf{f}$ is a pairtial elementary map in the sense of $\mathfrak{C}^K_K$.

iv. $\mathbf{f}(\overline{a}_1) = \overline{a}_2$.

v. For any $c, d \in \mathbf{K}$ such that $c, d \notin \mathrm{acl}_{\mathbf{f}}(B \cup K \cup \mathbf{\Gamma})$, there is an automorphism $\mathbf{g}$ of $\mathfrak{C}^K_{\Gamma,K}$ extending $\mathbf{f}$ and taking c to d.

Proof of the Claim: Firstly, add K to B, so assume $K \subseteq B$. Note that $\mathrm{card}(B) = \mathrm{card}(B \cup K)$ since $\mathrm{card}(K) = \aleph_0$. For each $\overline{b}$ from B, let $C_{\overline{b}}$ be the canonical base, which is finite [corollary 2.41], of $\mathrm{tp}^{\mathfrak{C}^K_K}(\overline{b}/\mathbf{\Gamma})$. Let $C = \bigcup_{\overline{b}\in^{<\omega}B} C_{\overline{b}}$. It is clear that $\mathrm{card}(C) = \mathrm{card}(B)$. Now, we prove that C satisfies the claim. Let $\overline{a}_1, \overline{a}_2$ from $\mathbf{\Gamma}$ such that $\mathrm{tp}^{\mathfrak{C}^\Gamma}(\overline{a}_1/C) = \mathrm{tp}^{\mathfrak{C}^\Gamma}(\overline{a}_2/C)$. By the lemma 1.28, Let $\mathbf{f}_1$ be an automorphism of $\mathfrak{C}^\Gamma$ fixing C and taking $\overline{a}_1$ to $\overline{a}_2$. Then, for any tuple $\overline{b}$ from B and any $\overline{a}$ from $\mathbf{\Gamma}$,

$$\mathrm{tp}^{\mathfrak{C}^K}(\overline{b}, \overline{a}/K) = \mathrm{tp}^{\mathfrak{C}^K}(\overline{b}, \mathbf{f}_1(\overline{a})).$$

Indeed, let $\phi(\overline{x}, \overline{y}) \in \mathrm{For}\, L_r(K)$. By the choice of C, there is some $\overline{c}$ from C and a formula $\psi(\overline{y}, \overline{c}) = \mathrm{d}_{\mathrm{tp}(\overline{b}/\mathbf{\Gamma},K)}\overline{x}\phi(\overline{y}) \in \mathrm{For}\, L_r(\overline{c})$ such that, for any $\overline{d}$

$$\mathfrak{C}^K_K \models \phi[\overline{b}, \overline{d}] \Leftrightarrow \mathfrak{C}^K_K \models \psi[\overline{d}, \overline{c}].$$

Now, $\psi[\mathfrak{B}_K] = X \subseteq A^m$ is a K-definable set in $(K,+,\cdot)$. Thus, $\underline{X \cap \Gamma^m}$ is an atomic formula in $\mathfrak{C}^\Gamma$. We have that $\mathfrak{C}^\Gamma \models \underline{X \cap \Gamma^m}[\overline{a}, \overline{c}]$ if and only if $\mathfrak{C}^\Gamma \models \underline{X \cap \Gamma^m}[\mathbf{f}_1(\overline{a}), \overline{c}]$, since $\mathbf{f}_1$ is an automorphism in $\mathfrak{C}^\Gamma$ and fixes C. Then, we have that $(\overline{a}, \overline{c}) \in \mathbf{X}$ if and only if $(\mathbf{f}_1(\overline{a}), \overline{c}) \in \mathbf{X}$. So

$$\phi(\overline{x}, \overline{y}) \in \mathrm{tp}^{\mathfrak{C}^K}(\overline{b}, \overline{a}/K) \Leftrightarrow \phi(\overline{x}, \overline{y}) \in \mathrm{tp}^{\mathfrak{C}^K}(\overline{b}, \overline{a}/K).$$

Define $\mathbf{f}_2: \ B \cup \mathbf{\Gamma} \to B \cup \mathbf{\Gamma}$ as $\mathbf{f}_2(b) = b$ and $\mathbf{f}_2(a) = \mathbf{f}_1(a)$ for any $a \in \mathbf{\Gamma}$ and $b \in B$. By the above observation, $\mathbf{f}_2$ is an elementary map in $\mathfrak{C}^K$. Let $\mathbf{f}$ be an elementary extension, in $\mathfrak{C}^K$, of $\mathbf{f}_2$ to $\mathrm{acl}_f(B \cup \mathbf{\Gamma})$. Now,

i. $\mathbf{f_1} \subseteq \mathbf{f}$ fixes C and $\mathbf{f}_2 \subseteq \mathbf{f}$ fixes B, so $\mathbf{f}$ fixes $B \cup C$ and $K \subseteq B$;

ii. $\mathbf{f}_{|\mathbf{\Gamma}} = \mathbf{f}_1$ is an automorphism;
iii. $\mathbf{f}$ is an elementary map in $\mathfrak{C}^K$;
iv. $\mathbf{f}(\overline{a}_1) = \mathbf{f}_1(\overline{a}_1) = \overline{a}_2$; and
v. given $c, d \notin \mathrm{acl}_f(B \cup \mathbf{\Gamma})$, it is a basic theorem of algebraically closed fields (and in monster models is still true) that there is an automorphism $\mathbf{g}$ in $\mathfrak{C}^K$ extending $\mathbf{f}$ and taking c to d, which is also an automorphism in $\mathfrak{C}^K_{\Gamma,K}$ since $\mathbf{g}$ fixes K and leaves $\mathbf{\Gamma}$ invariant.

Now, we conclude the proof of the theorem. By the claim, T_0 is ω-stable implies that T is ω-stable too, so T is totally transcendental [Theorem 2.18]. Indeed, let B such that $\mathrm{card}(B) = \aleph_0$. Then,

$$\begin{aligned}\mathbf{S}^{\mathfrak{C}^K_{\Gamma,K}}(B) =&\{\mathrm{tp}^{\mathfrak{C}^K_{\Gamma,K}}(d/B) \;:\; d \in \mathbf{K}\} = \\ =&\{\mathrm{tp}^{\mathfrak{C}^K_{\Gamma,K}}(d/B) \;:\; d \notin \mathrm{acl}_f(B \cup C \cup \mathbf{\Gamma})\} \cup \\ &\{\mathrm{tp}^{\mathfrak{C}^K_{\Gamma,K}}(d/B) \;:\; d \in \mathrm{acl}_f(B \cup C \cup \mathbf{\Gamma})\} = \\ =&I_1 \cup I_2,\end{aligned}$$

where C is the set given by the claim. Then, $\mathrm{card}(I_1) = 1$ and $\mathrm{card}(I_2) \leq \omega$. Indeed, it is clear that $\mathrm{card}(I_1) = 1$ by **v.** taking any $\overline{a}_1 = \overline{a}_2$. On the other hand,

$$I_2 = \bigcup_{\overline{a} \in^{<\omega} \mathbf{\Gamma}} \{\mathrm{tp}^{\mathfrak{C}^K_{\Gamma,K}}(d/B) \;:\; d \in \mathrm{acl}_f(B \cup C, \overline{a})\}.$$

If $\overline{a}_1$ and $\overline{a}_2$ have the same type in $\mathfrak{C}^\Gamma$ over C, let $\mathbf{f} : \mathrm{acl}_f(B \cup C \cup \mathbf{\Gamma}) \to \mathrm{acl}_f(B \cup C \cup \mathbf{\Gamma})$ be an elementary map given by the claim, then

$$\{\mathrm{tp}^{\mathfrak{C}^K_{\Gamma,K}}(d/B) \;:\; d \in \mathrm{acl}_f(B \cup C, \overline{a}_1)\} = \{\mathrm{tp}^{\mathfrak{C}^K_{\Gamma,K}}(d/B) \;:\; d \in \mathrm{acl}_f(B \cup C, \overline{a}_2)\}.$$

Hence,

$$I_2 = \bigcup_{p \in \mathbf{S}^{\mathfrak{C}^\Gamma}(C)} \{\mathrm{tp}^{\mathfrak{C}^K_{\Gamma,K}}(d/B) \;:\; d \in \mathrm{acl}_f(B \cup C, \overline{a}) \text{ where } \overline{a} \text{ realizes } p\}.$$

Since $\mathrm{card}(\mathbf{S}^{\mathfrak{C}^\Gamma}(C)) \leq \omega$ by ω-stability of T_0 and $\mathrm{card}(\mathrm{acl}_f(B \cup C, \overline{a})) \leq \omega$, we conclude that $\mathrm{card}(I_2) \leq \omega$.

Finally, we prove that $\mathbf{\Gamma}$ is one-based. By lemma 5.6 and theorem 3.22, it suffices to prove that every definable subclass of $\mathbf{\Gamma}^m$ in $\mathfrak{C}^K_{\Gamma,K}$ is definable in $\mathfrak{C}^\Gamma$. Note that any $\mathbf{\Gamma}^m$-definable subclass of $\mathbf{\Gamma}^m$ in $\mathfrak{C}^K_{\Gamma,K}$ is definable in $\mathfrak{C}^\Gamma$. So, let $\mathbf{X} \subseteq \mathbf{\Gamma}^m$ be a B-definable subclass in $\mathfrak{C}^K_{\Gamma,K}$ and assume B infinite. Let C be a set given by the claim. Then, $\mathbf{X}$ is C-definable in $\mathfrak{C}^K_{\Gamma,K}$. Indeed, if not, there are $\overline{a}_1 \in \mathbf{X}$ and $\overline{a}_2 \in \mathbf{X}$ such that $\mathrm{tp}^{\mathfrak{C}^K_{\Gamma,K}}(\overline{a}_1/C) = \mathrm{tp}^{\mathfrak{C}^K_{\Gamma,K}}(\overline{a}_2/C)$. Note that every definable class in $\mathfrak{C}^\Gamma$ is definable in $\mathfrak{C}^K_{\Gamma,K}$. Thus, $\mathrm{tp}^{\mathfrak{C}^\Gamma}(\overline{a}_1/C) = \mathrm{tp}^{\mathfrak{C}^\Gamma}(\overline{a}_2/C)$. Then, by the claim, there is an elementary map (in $\mathfrak{C}^K_{\Gamma,K}$) $\mathbf{f}$ taking $\overline{a}_1$ to $\overline{a}_2$ and fixing B and C, a contradiction since $\mathbf{X}$ is B-definable. $\square$

Remark. As in the case of the lemma 5.4, without the condition $|K| = \aleph_0$, the lemma is still true replacing totally transcendence by stable.

5.3 Hrushovski's proof in characteristic 0

The absolute Mordell-Lang's conjecture is not true in characteristic non zero. We give two counterexamples:

(1) Let $A/\overline{\mathbb{F}_p}$ be an abelian variety. All points of $A(\overline{\mathbb{F}_p})$ are torsion points. Since the Mordell-Lang's conjecture extends the Manin-Mumford's one, the Mordell-Lang's conjecture implies that any curve in an abelian variety is the translate of an elliptic curve. Of course, this is false, any smooth projective curve of genus different than 1 (and 0) is birrationaly embedded in its jacobian and is not the translate of an elliptic curve (see [5]).

(2) Let C be an smooth projective curve of genus $g > 1$ over $\mathbb{F}_p$, $A = \mathrm{Jac}(C)$, $F/\mathbb{F}_p$ an algebraically closed field different from $\overline{\mathbb{F}_p}$ and $K = \mathbb{F}_p(t)$ where $t \in C(F) \setminus C(\overline{\mathbb{F}_p})$. Let $\mathrm{Fr} : F \to F$ be the Frobenius' map, which acts on A and C since these are defined over $\mathbb{F}_p$. We know that $\Gamma = \{\mathrm{Fr}^n(t) : n \in \mathbb{N}\}$ is a finitely generated group. The absolute Mordell-Lang's conjecture says that $C(K) \cap \Gamma$ must be finite. However, $t \in C(K) \cap \Gamma$, so $\mathrm{Fr}^n(t) \in \Gamma \cap C(K)$ for every $n \in \mathbb{N}$. Since $t \notin C(\overline{\mathbb{F}_p})$, all these points are different (see pag. 208 of [14]).

Thus, we need a relative version of the Mordell-Lang's conjecture for positive characteristic:

Theorem 5.8. ***(Mordell-Lang's conjecture for function fields)*** *Let $k, K \models \mathrm{ACF}$ and $k \subseteq K$. Let X be an irreducible subvariety of an abelian variety A over K and $\Gamma \leq A(K)$ be a subgroup of finite rank. Suppose Stab_X is finite and $X \cap \Gamma$ is dense in X. Then, there are an abelian subvariety $B \leq A$ and an abelian variety S defined over K_0 and a subvariety $X_0 \subseteq S$ defined over K_0 too and a bijective morphism $h : B \to S$ such that $X = a_0 + h^{-1}(X_0)$ for a point $a_0 \in A$.*

We end this dissertation with an sketch of Hrushovski's proof of the relative Mordell-Lang's conjecture in characteristic 0.

Proof. **(Sketch)** (see [7])

The main idea is to replace Γ by a definable group. To do that, we add a derivation. Let δ be a derivation in K such that k be its field of constants and L be the differential closure of (K, δ), so k is 0-definable in L by the equation $\delta(x) = 0$. Since $L \models \mathrm{ACF}_0$, by completeness, we do not loss generality by replacing K by L. Let L' be an $\aleph_0$-saturated elementary extension of L and $k' = \underline{k}[L']$.

Step 1 It suffices to prove the statement for L' and k' instead of K and k, i.e., it suffices to prove that there are an abelian subvariety B of A, an abelian variety S' over k', a subvariety X_0' of S' defined over k' and a bijective morphism $h' : B \to S'$ such that $X = a_0' + {h'}^{-1}(X_0')$.

To simplify the notation, we may assume that $L = L'$. Write *δ-definable* for sets defined (with parameters) by formulas of the language of differential rings and *definable* for sets defined (with parameters) by formulas of the language of rings. Since $L \models \mathrm{ACF}$, L satisfies quantifier elimination for definable sets, i.e., every definable set in L is definable without quantifiers.

Step 2 Now, we replace $\Gamma \leq A$ by a δ-definable subgroup of A. Since Γ is a group of finite rank, there is a δ-definable subgroup $H \leq A$ containing Γ which has finite Morley rank.

Step 3 The main part of the proof is a technical argument showing that we may assume without loss of generality that H is an almost strongly minimal δ-definable connected group and no one-based.

Step 4 Finally, we conclude by showing that

i. there exists an abelian variety S defined over k and a bijective morphism f from $\overline{H}$ (the Zariski closure) to S such that $f(H) = S(k) = S \cap k^m$, and

ii. given $X \subseteq A$ subvariety defined over L such that $X \cap H$ is dense in X, there is a subvariety $X_0 \subseteq S$ defined over k such that $X = f^{-1}(X_0)$.

Indeed, let B be a strongly minimal δ-definable set such that $H \subseteq \mathrm{acl}(B)$. Then, B is a *Zariski geometry* and, since H is not one-based, B is not locally modular. Therefore, B interprets an algebraically closed field by the dichotomy theorem for Zariski geometries. Since B is strongly minimal, this algebraically closed field is δ-definably isomorphic to k. Therefore, $B \not\perp k$, so $H \not\perp k$. Then, by theorem theorem 3.28 and elimination of imaginaries, there is a δ-definable homomorphism $h: H \to G$ with finite kernel such that $G \subseteq \mathrm{dcl}(k)$ is a δ-definable group. Now, inverting the map h we obtain the map f that satisfies our claim.

A Axioms of set theory

Zermelo-Fraenkel-Skolem axioms.- The *Zermelo-Fraenkel-Skolem set theory* (ZFC) is the following theory in the language $L_{\mathrm{ZFC}} = \{\in\}$:

1. **Axiom of extensionality:** $\forall x \forall y (x = y \leftrightarrow \forall z (z \in x \leftrightarrow z \in y))$.
   [Two sets are equal if and only if they have the same elements].

2. **Axiom of pairing:** $\forall x \forall y \exists z \forall w (w \in z \leftrightarrow (w = x \vee w = y))$.
   [For any two sets x and y there exists the set $z = \{x, y\}$].

3. **Axiom schema of specification:** For any formula $\varphi(z, \overline{w})$ of L_{ZFE} and any variable y, non-free in $\varphi(z, \overline{w})$,

   $$\forall \overline{w} \forall x \exists y \forall z (z \in y \leftrightarrow (z \in x \wedge \varphi(z, \overline{w}))).$$

   [For any formula φ of L_{ZFE}, for every set x there exists the subset $y = \{z \in x \,:\, \varphi\}$].

4. **Axiom of union:** $\forall x \exists y \forall z (z \in y \leftrightarrow \exists w (w \in x \wedge z \in w))$.
   [For any set x, there exists the set $y = \bigcup x$].

5. **Axiom of power set:**

   $$\forall x \exists y \forall z (z \in y \leftrightarrow \forall w (w \in z \rightarrow w \in x)).$$

   [For any set x, there exists the set $y = \mathcal{P}(x)$].

6. **Axiom schema of replacement:** For any formula $\varphi(x', y', \overline{v})$ of L_{ZFE} and any variable y, non-free in φ,

   $$\forall \overline{v} (\forall x' \exists ! y' \varphi(x', y', \overline{v})) \rightarrow \forall x \exists y \forall z (z \in y \leftrightarrow \exists w \in x \ \varphi(w, z, \overline{v})).$$

   [For any formula φ of L_{ZFE}, if φ is a function formula, there exists the image set $y = \{z \,:\, \exists w (w \in x \wedge \varphi(w, z))\}$ by φ of x, for any set x].

7. **Axiom of infinity:** $\exists x (\emptyset \in x \wedge \forall y (y \in x \rightarrow y \cup \{y\} \in x))$.
   [There exists a set with infinitely many elements].

8. **Axiom of foundation:** $\forall x (\exists w \ w \in x \rightarrow \exists y (y \in x \wedge \forall z (z \notin x \vee z \notin y)))$.
   [In any non-empty set x there is an element y without common elements with x].

9. **Axiom of choice:**

   $$\forall x (x \neq \emptyset \rightarrow \exists f (f \text{ function} \wedge \forall z ((z \subseteq x \wedge z \neq \emptyset) \rightarrow \exists w \, ((z, w) \in f \wedge w \in z))))$$

   where "f function" represents the formula of L_{ZFE} expressing that f is a function and $(z, w) := \{\{z\}, \{z, w\}\}$.
   [For any non-empty set x there exists a function $f : \mathcal{P}(x) \setminus \{\emptyset\} \rightarrow x$ of choice, i.e., a function such that $f(y) \in y$ for every $y \in \mathcal{P}(x) \setminus \{\emptyset\}$].

If $\mathfrak{A} = (A, \in^{\mathfrak{A}})$ is a model of ZFC, a *set* in $\mathfrak{A}$ is an element of the universe A and a *class* in $\mathfrak{A}$ is a definable subset of A. It is clear that every set $a \in A$ is associated to a class given by $x \in a$, such that

$$x \in^{\mathfrak{A}} a \Leftrightarrow x \in \{x \in A \,:\, x \in^{\mathfrak{A}} a\}.$$

But, there are classes $\varphi(\mathfrak{A})$ such that there is not a set $a \in A$ satisfying $x \in \varphi(\mathfrak{A}) \Leftrightarrow x \in^{\mathfrak{A}} a$. These classes are named *proper classes*. The existence of this classes is a consequence of the Russel paradox. The *universal class* $\mathbb{V}$ is the proper class defined by $x = x$, and of course $\mathbb{V}$ is the universe of the model.

Neumann-Bernays-Gödel axioms.- The language L_{BGC} is a language of two sorts s_1 and s_2 extending L_{ZFC}. It consists in two binary relation symbols $\in_{(s_1,s_1)}$ and $\in_{(s_1,s_2)}$ —we use $\in$ for both. The variables of sort s_1 are the *sets* and we use small letters. The variables of sort s_2 are the *classes* and we use capital letters. The *Neumann-Bernays-Gödel set theory* (BGC) is the following L_{BGC}-theory:

1. **Axiom of extensionality:** $\forall x \forall y (x = y \leftrightarrow \forall z (z \in x \leftrightarrow z \in y))$.
2. **Axiom of pairing:** $\forall x \forall y \exists z \forall w (w \in z \leftrightarrow (w = x \vee w = y))$.
3. **Axiom of union:** $\forall x \exists y \forall z (z \in y \leftrightarrow \exists w (w \in x \wedge z \in w))$.
4. **Axiom of power set:**
$$\forall x \exists y \forall z (z \in y \leftrightarrow \forall w (w \in z \rightarrow w \in x)).$$
5. **Axiom of infinity:** $\exists x (\emptyset \in x \wedge \forall y (y \in x \rightarrow y \cup \{y\} \in x))$.
6. **Axiom of foundation:** $\forall x (\exists w \; w \in x \rightarrow \exists y (y \in x \wedge \forall z (z \notin x \vee z \notin y)))$.
7. **Axiom of extensionality for classes:**
$$\forall X \forall Y (X = Y \leftrightarrow \forall z (z \in X \leftrightarrow z \in Y)).$$
[Two classes are equal if and only if they have the same elements].
8. **Axiom schema of comprehension:** For any formula $\varphi(y, \overline{w}, \overline{W})$ of L_{BGC} where every variable of class sort is free and for any variable X, non-free in φ,
$$\forall \overline{w} \forall \overline{W} \exists X \forall y (y \in X \leftrightarrow \varphi(y, \overline{w}, \overline{W})).$$
[For every formula φ of L_{BGC} where every variable of class sort is free, there exists a class X whose elements are the sets satisfying φ].

9. **Axiom of replacement:**

$$\forall F(F \text{ function-class} \rightarrow \forall x \exists y \forall z(z \in y \leftrightarrow \exists w(w \in x \wedge (w,z) \in F)))$$

where "F function-class" represents the formula of L_{BGC} expressing that F is a function-class and $(w,z) := \{\{w\},\{w,z\}\}$.
[For every class F, if F is a function-class, then there exists the set image by F of x, for any set x].

10. **Axiom of global choice:**

$$\exists F(F \text{ function-class} \wedge \forall z(z \neq \emptyset \rightarrow \exists w((z,w) \in F \wedge w \in z)))$$

where "F function-class" represents the formula of L_{BGC} expressing that F is a function-class and $(z,w) := \{\{z\},\{z,w\}\}$. [There exists a function-class F of choice, i.e., a function-class such that $F(x) \in x$ for every $x \neq \emptyset$].

The axiom of replacement implies the axiom of specification:

$$\forall X(\exists x \forall y(y \in X \rightarrow y \in x) \rightarrow \exists z \forall w(w \in X \leftrightarrow w \in z)).$$

Let us prove it. If $X = \emptyset$, the statement is clear. If $X \neq \emptyset$ and $y_0 \in X$, consider the function-class defined by comprehension $F = \{(y,y) \ : \ y \in X\} \cup \{(y,y_0) \ : \ y \in x \wedge y \notin X\}$ and apply replacement with x.
By the axiom scheme of comprehension, we conclude that the axioms of replacement and specification of Neumann-Bernays-Gödel imply the axiom schemes of replacement and specification of Zermelo-Fraenkel-Skolem. Hence, BGE $\models$ ZFE.

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