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One-loop counterterms in first order quantum gravity

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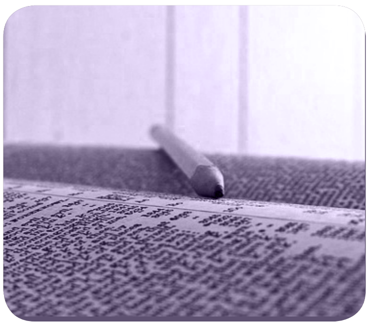

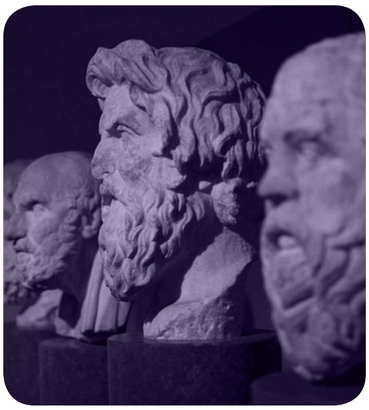

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One-loop counterterms in first order quantum gravity

Memoria del Trabajo de Fin de Máster realizada por

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October 2, 2017

Abstract

After an introduction to the background field technique and the heat kernel method, one loop counterterms are computed in first order formalism for the Einstein-Hilbert action using these tools. The obtained result coincides with the result found by 't Hooft and Veltman in second order formalism.

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1 Introduction

Since Albert Einstein introduced the theory of General Relativity (GR) in 1915, it has exhibited a great success in describing gravitational phenomena in a wide range of energies. Nevertheless, when the energies approach the Planck's mass, quantum effects become non-negligible leading to the need of a quantum theory of gravity. Unfortunately, the quantum version of GR has been found to be non renormalizable beyond one-loop (cf. [1] for a general review of quantum gravity and some approaches to the problem).

The first study of the renormalization of GR was carried out by 't Hooft and Veltman in a classic paper in 1973 [2], as a byproduct of their analysis of one-loop amplitudes in non-abelian gauge theories. Since the process of quantizing gauge theories was well understood, they studied the quantization of the gravitational theory as a gauge theory and treated it perturbatively using the background field technique. This technique, first introduced by DeWitt [3], allows for the quantization of the theory in an arbitrary background. To do this, an expansion in a small parameter κ is performed for all the fields in the theory, splitting them into their background value and a perturbation. This procedure simplifies the results, due to the fact that the gauge invariance of the background fields is mantained (only the perturbations have to be gauge fixed), restricting the set of counterterms that can appear to those with the gauge symmetry of the background.

The gravitational coupling is characterized by Newton's constant, which for three or more space-time dimensions has units of mass to a negative power. The ultraviolet behaviour of Einstein's theory of gravity is thus more singular than that of renormalizable theories. Indeed, elementary power-counting implies that, the only posible counterterms appearing have to be local invariants of dimensionalities increasing with each order of the perturbation expansion. Thus, the only hope for a successful perturbation expansion for Einstein's theory would require that, order by order, ultraviolet divergences cancel out of its S matrix. Therefore, finiteness rather than renormalizability, is the feature we have to require to a theory that includes gravitational interactions.

In this context, 't Hooft and Veltman [2] found that pure gravity (i.e. Einstein-Hilbert action without cosmological constant) turns out to be renormalizable to one-loop order, since the geometric invariants appearing in the counterterms are just R^2, $R_{\mu\nu}R^{\mu\nu}$ and $R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma}$. In four dimensions, the last invariant turns out to be a topological invariant so it can be written in terms of the first two invariants that precisely cancel on-shell. The computation of the two-loop counterterm was carried out by Goroff and Sagnotti [4], finding that quantum gravity is already divergent at two-loop order. In this case there is a surviving geometrical invariant even on-shell. The invariants appearing are all proportional to R or $R_{\mu\nu}R^{\mu\nu}$ so that they cancel on-shell, but one of them takes the form $R_{\alpha\beta}{}^{\gamma\delta}R_{\gamma\delta}{}^{\epsilon\zeta}R_{\epsilon\zeta}{}^{\alpha\beta}$ and does not vanish even when imposing the background equations of motion. When the simplest type of matter is included in the theory, such as a scalar field, the theory is no longer finite even at one-loop order as also found by 't Hooft and Veltman.

The need of a quantum theory of gravity that is complete in the ulraviolet has lead the way to a number of possible solutions to the problem of the non renormalizability of the theory. One of the solutions is to introduce higher curvature terms. It was found that quadratic theories of gravity are indeed renormalizable [5]. However, the prize to pay is that they are not unitary, hence making these class of theories equally useless a priori, although efforts have been made in order to fix the problem [6].

On the other hand, it is well-known that when considering the Palatini version of the Einstein-Hilbert Lagrangian, that is, the first order formalism for the classical case (i.e. the metric and the connection are treated independently), the connection is required to be the Levi-Civita one once the equations of motion are imposed. However, when more general quadratic in curvature metric-affine actions are considered in first order, this relationship disappears even on shell, hence allowing for more general connections. That is, the equations of motion do not force the connection to be the Levi-Civita one. This is of particular interest because when analyzing quadratic theories in first order formalism, there are no propagators falling down faster than $\frac{1}{p^2}$, indicating that there is still room for the theory to be unitary. The study of quadratic theories of gravity in first order formalism, being renormalizable, could give rise to a unitary and renormalizable theory of gravity [7].

In this scenario, the computation to one loop order for the Einstein-Hilbert case in first order formalism paves the way for future studies of more complex theories. The aim of this work is to compute the quantum corrections to the gravitational action to first order in the coupling constant, κ, using the background field method as 't Hooft and Veltman did, but in first order formalism, where the fields $g_{\mu\nu}$ and $\Gamma^{\lambda}_{\alpha\beta}$ are independent. This problem was studied by Buchbinder and Shapiro [8], but the result obtained did not match the one 't Hooft and Veltman obtained in second order formalism (even with the same field parametrizations). The exact coefficients of the counterterms are highly dependent on the parametrization used for the different fields appearing in the theory and also of the gauge fixing used in the computations, so for this work we take exactly the parametrization used by 't Hooft and Veltman and also the same gauge fixing. Our results [9] are in contradiction with [8] but coincide with the result of 't Hooft and Veltmann found in second order formalism.

Another difference from the computation in second order formalism carried out by 't Hooft and Veltman, is that our computation relies on the heat kernel technique in order to extract the counterterms. The heat kernel technique (cf. [10, 11] for a general review of the subject) is mostly used in one-loop computations and it allows to extract the infinite part of the effective action (i.e the generator of the 1PI Green's functions). This technique makes the calculation of the counterterms possible without having to go through all the one-loop diagrams of the theory. In this work we combine the background field tecnique and the heat kernel method. In the former the computation of the effective action turns out to be analogous to the computation of the determinant of certain operator and then using the latter, one can extract the divergent piece of that determinant. This simplifies enormously the computations first carried by 't Hooft and Veltman, where their approach was diagramatic.

Let us set up the problem we are going to study. The action we are considering is the usual Einstein-Hilbert action without cosmological constant or any matter coupled to the gravitational field

$$S_{\text{EH}} \equiv -\frac{1}{2\kappa^2}\int d^n x\sqrt{|g|}g^{\mu\nu}R_{\mu\nu} \tag{1}$$

where $\kappa^2 = 8\pi G$, and we assume that speed of light $c = 1$.

Throughout this work we follow the Landau-Lifshitz spacelike conventions, in particular

$$R^{\mu}{}_{\nu\rho\sigma} = \partial_\rho \Gamma^{\mu}_{\nu\sigma} - \partial_\sigma \Gamma^{\mu}_{\nu\rho} + \Gamma^{\mu}_{\lambda\rho}\Gamma^{\lambda}_{\nu\sigma} - \Gamma^{\mu}_{\lambda\sigma}\Gamma^{\lambda}_{\nu\rho} \ . \tag{2}$$

We define the Ricci tensor as

$$R_{\mu\nu} \equiv R^{\lambda}{}_{\mu\lambda\nu} \ , \tag{3}$$

and the commutator with our conventions is

$$[\nabla_\mu, \nabla_\nu] h^{\alpha\beta} = h^{\beta\lambda} R^{\alpha}{}_{\lambda\mu\nu} + h^{\alpha\lambda} R^{\beta}{}_{\lambda\mu\nu} \ . \tag{4}$$

The important thing we have to keep in mind is that in first order formalism the metric and the connection are independent and hence there is no fixed relation between them.

The outline of this work is as follows. In the second section, we focus on the introduction of the background field technique and show its use in one-loop order computations. The third section is devoted to the understanding of the heat kernel technique and to the computation of the coefficients appearing in the formula that will give us the divergent piece of the determinant, and thus, the counterterms. In the fourth section, we perform the explicit computation of the one-loop counterterms in first order formalism for the Einstein-Hilbert action without cosmological constant. Finally some conclusions are left for section 5.

2 Background field method

An essential tool for this analysis is the background field method, first introduced by DeWitt [3]. The background field method is a technique for quantizing gauge field theories without losing explicit gauge invariance completely.

The starting step in any gauge theory is, of course, a gauge invariant Lagrangian. However, when quantizing the theory a gauge fixing must be imposed. This leads to a manifestly non gauge invariant Lagrangian, consisting of the original Lagrangian plus gauge-fixing and ghost terms. Nevertheless, any computed physical quantity will be gauge invariant and independent of the particular gauge choice. The problem is that quantities with no direct physical interpretation, such as off-shell Green's functions or divergent counterterms, will not be in general gauge invariant. In the background field approach, the key point is that explicit gauge invariance, present in the original Lagrangian, is still avalaible once gauge-fixing and ghost terms have been added. As a result, in this formalism, even the unphysical quantities we were discussing take a gauge invariant form.

First of all, we start by introducing the functional approach of the path integral. The final goal of any field theory computation is to obtain the S matrix, from which we can compute the amplitude of any process we are interested in. The S matrix can be obtained from the Green's functions of the theory by LSZ reduction. In the functional approach, the Green's functions are determined by taking functional derivatives with respect to the source function J of the generating functional

$$Z[J] = \int \mathcal{D}A \; e^{i(S[A]+JA)}. \qquad (5)$$

However it is more practical to work only with connected Green's functions, as the disconnected pieces do not contribute to the S matrix. These are generated by taking J derivatives of

$$W[J] = -i \ln Z[J] \;, \qquad (6)$$

where W is called the free energy.

The connected Green's functions can be further simplified by expressing them in terms of one-particle-irreducible (1PI) pieces. This turns out to be useful for the computation since it is easier to actually get the 1PI Green's functions and then combine them into the diagrams we want to compute, instead of computing explicitly all connected diagrams. The 1PI Green's functions are generated by a functional called the effective action Γ. It is defined as

$$\Gamma[A_c] = W[J] - JA_c \;, \qquad (7)$$

where

$$A_c \equiv \frac{\delta W}{\delta J} \;. \qquad (8)$$

After having seen this, we understand that the effective action $\Gamma[A_c]$ is an important quantity to compute in a field theory. Once this is known, the S matrix can be constructed after joining together trees of 1PI parts to generate the full connected Green's functions. The background field method is a convenient way of computing the effective action. Unfortunately, it is not possible in general to compute exactly the effective action. As we will see in the next section, we can only compute its divergent part to some order in the computation and use it to extract the counterterms to that order.

To see how this method works, we start by expanding the generic field $\tilde{A}_\mu$ around an arbitrary background configuration

$$\tilde{A}_\mu = \bar{A}_\mu + A_\mu \;, \qquad (9)$$

where A is now a perturbation around the background, also called the quantum field. In the functional integral we are going to integrate over quantum fields only (otherwise nothing changes), so that we have

$$Z[\bar{A}] = \int \mathcal{D}A_\mu e^{iS[\bar{A}+A]} \qquad (10)$$

The background is assumed to obey the classical equations of motion, namely

$$\frac{\delta S}{\delta \tilde{A}_\mu}\Big|_{\bar{A}_\mu} = 0 \;. \qquad (11)$$

So that at the one-loop order, we will have

$$\begin{aligned} e^{iW[\bar{A}]} &= \int \mathcal{D}A \exp\left\{ iS[\bar{A}] + i\int d(vol)\; \frac{1}{2} AK[\bar{A}]A + i\int d(vol)\; JA \right\} = \\ &= \exp\left\{ iS[\bar{A}] - i\frac{1}{2}\ln\det K[\bar{A}] + i\frac{1}{2}\int d(vol)\; JK^{-1}[\bar{A}]J \right\} \;, \end{aligned} \qquad (12)$$

where in the last line we have used that the gaussian integral over A can be computed by the usual technique of completing squares

$$\frac{1}{2}AKA + JA = \left[A + \frac{1}{\sqrt{2}}\ JK^{-1}1\right] K \left[A + \frac{1}{\sqrt{2}}\ K^{-1}J\right] \ . \tag{13}$$

Here we use that the measure $\mathcal{D}A$ is translational invariant so that the result of the integral is the well known $(\det K)^{-1/2}$. After this integration we obtain then

$$\begin{aligned} &A_c = K^{-1}[\bar{A}]J \ , \\ &J = K[\bar{A}]A_c \ , \end{aligned} \tag{14}$$

and therefeore

$$\begin{aligned} \Gamma^{BF}[A_c, \bar{A}] &= W[J(A_c)] - \int d(vol)\ JA_c = \\ &= S[\bar{A}] - \frac{1}{2}\ln\det K[\bar{A}] - \frac{1}{2}\int d(vol)\ A_c K[\bar{A}]A_c \ . \end{aligned} \tag{15}$$

Larry Abbott [12] was able to prove that the effective action computed by the background field method is related to the ordinary effective action. To do this, we shift the variable of integration A by taking $A \to A - \bar{A}$. Doing this, we obtain that

$$Z^{BF}[J, \bar{A}] = Z[J]e^{-i\bar{A}J} \quad \longrightarrow \quad W^{BF}[J, \bar{A}] = W[J] - J\bar{A} \ , \tag{16}$$

and differentiating with respect to J

$$A_c^{BF} = A_c - \bar{A} \ . \tag{17}$$

Finally the total effective action is

$$\Gamma^{BF}[A_c^{BF}, \bar{A}] = W[J] - J\bar{A} - JA_c + J\bar{A} = \Gamma[A_c] \ , \tag{18}$$

and recalling $A_c = A_c^{BF} + \bar{A}$ we get

$$\Gamma^{BF}[A_c^{BF}, \bar{A}] = \Gamma[A_c^{BF} + \bar{A}] \ . \tag{19}$$

As a special case we get

$$\Gamma[A_c] = \Gamma^{BF}[0, \bar{A} = A_c] \ . \tag{20}$$

Coming back to the case we were analizying for one loop, we finally get

$$\Gamma[A_c] = \Gamma^{BF}[0, \bar{A} = A_c] = W[\bar{A}] = S[\bar{A}] - \frac{1}{2}\ln\det K[\bar{A}] \ , \tag{21}$$

so that we see that we can obtain the one-loop effective action as the background field free energy. Also, we notice that the effective action only depends on a determinant which is a function of the background field, so that if we maintain the gauge invariance of the background, the counterterms coming from the infinite part of the determinant will be gauge invariant and hence we can restrict them enormously.

Focusing in the gauge invariance, for the total action we have that the gauge transformations can be written as

$$(\bar{A}_\mu + A_\mu)' = g(\bar{A}_\mu + A_\mu)g^{-1} + g\partial_\mu g^{-1} \ , \tag{22}$$

where g is an element of the gauge group acting on the fields (we use this notation for simplicity).

From these tranformations there is a subset called quantum gauge transformations which are those under which the background field remains unchanged

$$\begin{aligned} \bar{A}'_\mu &= \bar{A}_\mu \ , \\ A'_\mu &= g(\bar{A}_\mu + A_\mu + \partial_\mu)g^{-1} - \bar{A}_\mu \ . \end{aligned} \tag{23}$$

We want to gauge fix these transformations. The key point of this method is that there is still another set of transformations, the background gauge transformation, which can be kept even when gauge fixing the quantum ones

$$\begin{aligned} \bar{A}'_\mu &= g(\bar{A}_\mu + \partial_\mu)g^{-1} \ , \\ A'_\mu &= gA_\mu g^{-1} \ . \end{aligned} \tag{24}$$

This is the main advantage of the background field method, that is, the gauge invariance of the background field can be kept during the calculations.

In the next section we will use the heat kernel technique to compute the divergent piece of the determinant which gives the effective action.

3 Heat kernel

There is a very powerful set of computational techniques, related to the zeta function regularization, which is known as the heat kernel approach [10, 11],because the heat equation plays a significant role in it. Its power becomes apparent when combined with the background field method, as we are doing in this computation.

We begin with the divergent integral first studied by Schwinger [13]

$$I(\lambda) = \int_0^\infty \frac{dx}{x} e^{-x\lambda} \ . \tag{25}$$

This integral diverges so it has to be regularized. We can define it as

$$I(\lambda) \equiv \lim_{\epsilon\to 0} I(\epsilon,\lambda) \equiv \lim_{\epsilon\to 0} \int_\epsilon^\infty \frac{dx}{x} e^{-x\lambda} \ . \tag{26}$$

With this definition we find the following

$$\lim_{\epsilon\to 0} \frac{\partial}{\partial\lambda} I(\epsilon,\lambda) = -\frac{1}{\lambda} \quad \longrightarrow \quad I(\lambda) = -\ln\lambda + C \ . \tag{27}$$

We can now define the determinant of an operator with discrete eigenvalues as

$$\ln \det \Delta = \text{Tr } \ln \Delta \equiv \sum_n \ln \lambda_n \ , \tag{28}$$

where λ_n are the eigenvalues of the operator. Given an operator (with purely discrete, positive spectrum) we can generalize the above idea an write

$$\ln \det \Delta \equiv - \int_0^\infty \frac{d\tau}{\tau} \text{tr } e^{-\tau \Delta} \ . \tag{29}$$

The trace here involves not only discrete indices, but also includes an space-time integral. We have introduced τ, called the proper time [1], as this is related to the proper time method introduced by Schwinger [13]. The formula we have introduced for the determinant is a purely formal formula and, as we can see, ultraviolet divergences appear as we approach the lower limit of the integral.

Let us define now the heat kernel associated to that operator as

$$K(\tau) \equiv e^{-\tau \Delta} \ , \tag{30}$$

which obeys the heat equation

$$\left(\frac{\partial}{\partial \tau} + \Delta \right) K(\tau) = 0 \ . \tag{31}$$

For small proper time, there exists an expansion first introduced by Schwinger in [13] and then reformulated in geometric language and extended to curved spaces by DeWitt in [3], which allows for an expansion of the heat kernel as a Taylor series

$$K\left(\tau; x, y\right) = K_0\left(\tau; x, y\right) \sum_{p=0}^{\infty} a_p\left(x, y\right) \tau^p \ . \tag{32}$$

Here K_0 is the solution of the simplest operator we can have, namely the flat Laplacian, so that it has the form

$$K_0(x, y; \tau) = \frac{1}{(4\pi\tau)^{n/2}} e^{\frac{-\sigma(x,y)}{2\tau}} \ , \tag{33}$$

where $\sigma(x, y)$ is the Synge's world function which in flat space happens to be $\sigma(x, y) = \frac{1}{2}(x - y)^2$, and in general it is related to the geodesic distance. For the purposes of this work we only need to know that the limit $\sigma(x, y) \to 0$ is equivalent to $x \to y$.

The coefficients $a_p\left(x, y\right)$ characterize the operator whose determinant is to be computed. Actually, for our purpose, only their diagonal part, $a_p\left(x, x\right)$ is relevant. Using the Schwinger-DeWitt expansion in the definition of the determinant we get

[1] As it can be seen τ has no units of time but it is related to the proper spacte-time interval (and thus has units of distance squared) and it is called Schwinger's proper time for historical reasons.

$$\log\det\Delta = -\int_0^\infty \frac{d\tau}{\tau}\mathrm{Tr}\,K(\tau) \equiv -\lim_{\sigma\to 0}\int_0^\infty \frac{d\tau}{\tau}\frac{1}{(4\pi\tau)^{n/2}}\int d(vol)\sum_{p=0}^{\infty}\tau^p\,\mathrm{Tr}\,a_p(x,y)\,e^{\frac{-\sigma(x,y)}{2\tau}}\ , \tag{34}$$

where in the last step we have regularized the divergent integral by point splitting, that is, we insert also the non diagonal part $a_p(x,y)$ of the coefficients and use $\sigma(x,y)$ as a regulator.

All ultraviolet divergences are encoded in the region $\tau \sim 0$, and this is indeed all the information we can obtain from the determinant as we have already performed an expansion around small proper times. Changing the order of integration and performing first the proper time integral we get

$$\log\det\Delta = -\int d^n x\ \sqrt{|g|}\ \lim_{\sigma\to 0}\sum_{p=0}^{\infty}\frac{\sigma\,(x,y)^{p-n/2}}{4^p\pi^{n/2}}\ \Gamma\left(\frac{n}{2}-p\right)\ \mathrm{tr}\,a_p\,(x,y)\ . \tag{35}$$

We can get the one-loop order divergence for $n=4$ by taking $p=2$ and using dimensional regularization $n=4-\epsilon$, obtaining

$$\frac{1}{2}\log\det\Delta = -\frac{1}{4\pi^2}\frac{1}{n-4}\int d^n x\ \sqrt{|g|}\ \mathrm{tr}\,a_2\,(x,x)\ , \tag{36}$$

where we have used

$$\begin{aligned}\Gamma\left(-\frac{\epsilon}{2}\right) &\sim -\frac{\epsilon}{2}-\gamma_E+\mathcal{O}(\epsilon)\ ,\\ \sigma(x,y)^{\epsilon/2} &\simeq 1+\frac{\epsilon}{2}\ln\sigma(x,y)+\mathcal{O}(\epsilon^2)\ ,\end{aligned} \tag{37}$$

and we do not take into account the constant terms, but just the divergent piece. As already mentioned, When taking the limit $\sigma\to 0$ we are taking the limit $x\to y$ so that we pick the diagonal part of the coeffcients.

After doing this, we need to compute the coefficient a_2. In order to do that, we are going to study general properties of the heat kernel for Laplace type operators and also use the consistency of the method [11]. We consider a smooth compact Riemannian manifold $\mathcal{M}$ of dimension n with a vector bundle $\mathcal{V}$ defined over $\mathcal{M}$. If we have second order operators of Laplace type we can write them as

$$D = -(g^{\mu\nu}\partial_\mu\partial_\nu + a^\sigma\partial_\sigma + b)\ , \tag{38}$$

where a and b are matrix valued functions in $\mathcal{M}$.

There is a unique endomorphism E (another matrix valued function) allowing us to rewrite the operator as

$$D = -(g^{\mu\nu}\nabla_\mu\nabla_\nu + E)\ , \tag{39}$$

where ∇_μ is the usual covariant derivative and

$$\begin{aligned}E &= b - g^{\nu\mu}(\partial_\mu\omega_\nu + \omega_\nu\omega_\mu - \omega_\sigma\Gamma^\sigma_{\nu\mu})\ ,\\ \omega_\delta &= \frac{1}{2}g_{\nu\delta}(a^\nu + g^{\mu\sigma}\Gamma^\nu_{\mu\sigma}\ \mathcal{I}_\mathcal{V})\ .\end{aligned} \tag{40}$$

The operators we are analyzing here are called minimal operators because they are the set of operators that have trivial internal index structure in the term cointainig second deriatives, meaning that they have that this term is just contracted with the metric. The determinant of such operators can be computed with the usual heat kernel techniques [10, 11].

The operator $\exp(-tD)$ with possitive t is trace class on the space of square integrable functions $\mathrm{L}^2(\mathcal{V})$, that is, for an smooth funcion f we can write

$$K(t, f, D) = \mathrm{Tr}_{\mathrm{L}^2}(f\ \exp(-tD))\ , \tag{41}$$

which is the definition of the heat kernel of the operator D, as we have already studied above. We can expand this definition and write

$$K(t, f, D) = \int_{\mathcal{M}} d^n x \sqrt{g}\ \mathrm{Tr}_{\mathcal{V}}\ K(t; x, x, D)\ f(x)\ , \tag{42}$$

where the trace is taken over the internal indices and $K(t; x, x; D)$ is an $y \to x$ limit of the solution to the heat equation with a certain initial condition

$$\begin{aligned} &(\partial_t + D_x) K(t; x, y, D) = 0\ , \\ &K(0; x, y, D) = \delta(x, y)\ . \end{aligned} \tag{43}$$

If we are working with manifolds without boundaries (or manifolds with boundaries in which the fields are subject to certain special boundary conditions), there is an asymptotic expansion as $t \to 0$ as we saw earlier

$$K(t, f, D) = \mathrm{Tr}_{\mathrm{L}^2}(f\ \exp(-tD)) \simeq \sum_{k \geq 0} t^{(k-n)/2} a_k(f, D)\ , \tag{44}$$

where the coefficients of the expansion are called the small proper time expansion heat kernel coefficients and are defined as

$$a_k(f, D) = (4\pi)^{-n/2} \int_{\mathcal{M}} d^n x\ \sqrt{g}\ a_k(x, x)\ f(x)\ . \tag{45}$$

There are two ways of computing the heat kernel coefficients $a_k(f, D)$. The first one is the DeWitt iterative procedure. The second way, and the one we are following in this case, is Gilkey's method [14]. We start with very general properties of the heat kernel coefficients, and for simplicity we consider a smooth compact Riemannian manifold $\mathcal{M}$ without boundary. We take an operator of Laplace type D on $\mathcal{V}$ and a smooth function f. Analyzing again the asymptotic expansion around small proper time, there are two statements that the coefficients has to fulfill

- All coefficients with odd index k vanish $\longrightarrow a_{2j+1} = 0$
- All the remaining coefficients $a_{2j}(f, D)$ can be computed in terms of geometric invariants of the right dimension

Going deeper in these statements, the second one means that we can write the heat kernel coefficients as integrals of the right dimensional geometric invariants, giving

$$a_k(f,D) = \mathrm{Tr}_{\mathcal{V}} \int_{\mathcal{M}} d^n x \, \sqrt{g} \, \{ f(x) a_k(x;D) \} = \sum_I \mathrm{Tr}_{\mathcal{V}} \int_{\mathcal{M}} d^n x \, \sqrt{g} \, \left\{ f u^I \mathcal{A}_k^I(D) \right\} \ , \quad (46)$$

where as anticipated, $\mathcal{A}_k^I(D)$ are all possible independent invariants of dimension k constructed from E, Ω, $R_{\mu\nu\rho\sigma}$ and their derivatives (we do not have other relevant quantities in our theory). For example, taking $k = 2$ it is easy to check that we can only construct two geometric invariants, namely E and R.

On the other hand, the first statement can be understood easily as one cannot construct an odd-dimensional invariant without including an odd number of derivatives, as they are the only objects that have odd dimension. To respect geometric invariance, that is, general diffeomorphism invariance (as the operator whose heat kernel we want to compute does), the derivatives have to appear in pairs. Hence, the odd coefficients vanish as no invariant with odd dimension can be included in manifolds without boundary.

We continue considering a product of two manifolds, $\mathcal{M}_1$ and $\mathcal{M}_2$, with coordinates x_1 and x_2 respectively and, $D = D_1 \otimes D_2$, so that

$$\exp(-tD) = \exp(-tD_1) \otimes \exp(-tD_2) \ . \quad (47)$$

If we multiply both equations by $f(x_1, x_2) = f_1(x_1) f_2(x_2)$ and take the functional traces as defined in (44) we get

$$\sum_{k\geq 0} t^{(k-n)/2} a_k(f,D) = \sum_{p\geq 0} t^{(p-n)/2} a_p(f_1(x_1),D) \sum_{q\geq 0} t^{(q-n)/2} a_q(f_2(x_2),D) \ . \quad (48)$$

We can find that for each k we have the relation

$$a_k(x,D) = \sum_{p+q=k} a_p(x_1,D) a_q(x_2,D) \ . \quad (49)$$

This allows us to find a relation between the coefficients u^I. For instance, if we take $\mathcal{M}_1 = S^1$ with $0 < x_1 \leq 2\pi$ and $D_1 = \nabla_{x_1}^2$, then all geometric invariants are defined just by the D_2 part and are independent of x_1, so that

$$\begin{aligned} a_k(f(x_2),D) &= \int_{S^1\times\mathcal{M}_2} d^n x \, \sqrt{g} \, \sum_I \mathrm{Tr}_{\mathcal{V}} \left\{ f(x_2) u^I(n) \mathcal{A}_k^I(D) \right\} \\ &= 2\pi \int_{\mathcal{M}_2} d^{n-1} x \, \sqrt{g} \, \sum_I \mathrm{Tr}_{\mathcal{V}} \left\{ f(x_2) u^I(n) \mathcal{A}_k^I(D_2) \right\} \ , \end{aligned} \quad (50)$$

where the geometric invariants we have to compute only depend on D_2 as we already mentioned. We can also use (49). For that, we know that the eigenvalues of the operator D_1 are l^2 with $l \in \mathbb{Z}$. In this way, we can compute the heat kernel expansion for D_1 using

Poisson sumation formula[2]:

$$K(t,D_1)=\sum_{l\in\mathbb{Z}}\exp(-tl^2)=\sqrt{\frac{\pi}{t}}\,\exp(-\pi^2l^2/t)\simeq\sqrt{\frac{\pi}{t}}+\mathcal{O}(e^{-1/t})\ . \tag{52}$$

Since we are looking for the coefficients in the small proper time expansion, exponentially supressed terms have no effect on the heat kernel coefficients, and the only non-zero coefficient is $a_0(1,D_1)=\sqrt{\pi}$. Therefore, substituing in (49)

$$a_k(f(x_2),D)=\sqrt{\pi}\int_{\mathcal{M}_2}d^{n-1}x\,\sqrt{g}\,\sum_I \mathrm{Tr}_{\mathcal{V}}\left\{f(x_2)\,u^I_{(n-1)}\mathcal{A}^I_n(D_2)\right\}\ . \tag{53}$$

Comparing (50) and (53) we get

$$u^I_{(n)}=\sqrt{4\pi}u^I_{(n+1)}\ , \tag{54}$$

so that

$$u^I_{(n+1)}=(4\pi)^{-1/2}u^I_{(n)}=(4\pi)^{-n/2}u^I_{(0)}\ , \tag{55}$$

and therefore the only dependence on the dimension n is via an overall normalization factor $(4\pi)^{-n/2}$.

To go further in the computation of the heat kernel coefficients we need some variational equations

$$\frac{d}{d\epsilon}\Big|_{\epsilon=0}a_k(1,\exp^{-2\epsilon f}D)=(n-k)a_k(f,D)\ , \tag{56}$$

$$\frac{d}{d\epsilon}\Big|_{\epsilon=0}a_k(1,D-\epsilon F)=a_{k-2}(F,D)\ , \tag{57}$$

$$\frac{d}{d\epsilon}\Big|_{\epsilon=0}a_{n-2}(\exp^{-2\epsilon f}F,\exp^{-2\epsilon f}D)=0\ , \tag{58}$$

where f and F are smooth functions. The derivation of these equations is carried out in (A.1).

Next, we write down a general expression for a_k containing all invariants $\mathcal{A}^I_k$ of dimension k with arbitrary coefficient u^I. As we already saw, we can rescale all the constants by $(4\pi)^{-n/2}$ and call the other constants α_I, which now are independent of n. For the purposes of this work, we just need to compute the $a_4(f,D)$ coefficient (often it is also called a_2 because it is the third lowest non zero coefficient), so that we write down

[2] The Poisson summation formula can be written as

$$\sum_{-\infty}^{\infty}h(2k\pi)=\frac{1}{2\pi}\sum_{-\infty}^{\infty}\int_{-\infty}^{\infty}dy\ h(y)e^{-iky} \tag{51}$$

the expressions for the three lowest coeffcients

$$
\begin{aligned}
a_0(f,D) &= (4\pi)^{-n/2}\int_{\mathcal{M}} d^nx\,\sqrt{g}\,\mathrm{Tr}_{\mathcal{V}}\left\{\alpha_0 f\right\}\ , & (59)\\
a_2(f,D) &= (4\pi)^{-n/2}\frac{1}{6}\int_{\mathcal{M}} d^nx\,\sqrt{g}\,\mathrm{Tr}_{\mathcal{V}}\left\{f(\alpha_1 E+\alpha_2 R)\right\}\ , & (60)\\
a_4(f,D) &= (4\pi)^{-n/2}\frac{1}{360}\int_{\mathcal{M}} d^nx\,\sqrt{g}\,\mathrm{Tr}_{\mathcal{V}}\big\{f(\alpha_3 E_{;\mu}^{;\mu}+\alpha_4 RE+\alpha_5 E^2+\alpha_6 R_{;\mu}^{;\mu}+\\
&+ \alpha_7 R^2+\alpha_8 R_{\mu\nu}R^{\mu\nu}+\alpha_9 R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma}+\alpha_{10}\Omega_{\mu\nu}\Omega^{\mu\nu})\big\}\ . & (61)
\end{aligned}
$$

- The coefficient α_0 follows from the heat kernel expansion of the scalar Laplacian on S^1 (already computed in (52)), so that $\boxed{\alpha_0 = 1}$.
- Now we use (57) with $k = 2$

$$
\begin{aligned}
\frac{d}{d\epsilon}\Big|_{\epsilon=0} a_2(1, D-\epsilon F) &= a_0(F,D) \longrightarrow\\
\frac{1}{6}\int_{\mathcal{M}} d^nx\,\sqrt{g}\,\mathrm{Tr}_{\mathcal{V}}\left\{\alpha_1 F\right\} &= \int_{\mathcal{M}} d^nx\,\sqrt{g}\,\mathrm{Tr}_{\mathcal{V}}\left\{F\right\}\ , \qquad (62)
\end{aligned}
$$

  and we can directly extract $\boxed{\alpha_1 = 6}$.

- Taking now (57) for $k = 4$

$$
\begin{aligned}
\frac{d}{d\epsilon}\Big|_{\epsilon=0} a_4(1, D-\epsilon F) &= a_2(F,D) \longrightarrow \qquad (63)\\
\frac{1}{360}\int_{\mathcal{M}} d^nx\sqrt{g}\,\mathrm{Tr}_{\mathcal{V}}\left\{\alpha_4 RF+2\alpha_5 EF\right\} &= \frac{1}{6}\int_{\mathcal{M}} d^nx\sqrt{g}\,\mathrm{Tr}_{\mathcal{V}}\left\{\alpha_1 FE+\alpha_2 FR\right\}\ .
\end{aligned}
$$

  From this equation we get $\boxed{\alpha_5 = 180}$ and $\boxed{\alpha_4 = 60\alpha_2}$.

To proceed further we have to take into account local scale transformations defined in (56) and (58). We start with the transformations

$$
\begin{aligned}
&\tilde{L} = e^{-2\epsilon f}L\ ,\\
&\tilde{g}^{\mu\nu} = e^{-2\epsilon f}g^{\mu\nu}\ ,\\
&\tilde{a}^{\mu} = e^{-2\epsilon f}a^{\mu}\ ,\\
&\tilde{b} = e^{-2\epsilon f}b\ . \qquad (64)
\end{aligned}
$$

With this transformations we can deduce how the other relevant quantities transform. This is analyzed in (A.2) and it can be seen that we can get the following useful relations[3]

$$\left.\frac{d}{d\epsilon}\right|_{\epsilon=0} \sqrt{\tilde{g}} = nf\sqrt{g} \ , \tag{65}$$

$$\left.\frac{d}{d\epsilon}\right|_{\epsilon=0} \tilde{R}_{\mu\nu\rho\sigma} = -2fR_{\mu\nu\rho\sigma} + \delta_{\nu\sigma}f_{;\mu\rho} + \delta_{\mu\rho}f_{;\nu\sigma} - \delta_{\mu\sigma}f_{;\nu\rho} - \delta_{\nu\rho}f_{;\mu\sigma} \ , \tag{66}$$

$$\left.\frac{d}{d\epsilon}\right|_{\epsilon=0} \tilde{E} = -2fE + \frac{1}{2}(n-2)f_{;\mu}^{;\mu} \ , \tag{67}$$

$$\left.\frac{d}{d\epsilon}\right|_{\epsilon=0} \tilde{R} = -2fR - 2(n-1)f_{;\mu}^{;\mu} \ , \tag{68}$$

$$\left.\frac{d}{d\epsilon}\right|_{\epsilon=0} \tilde{E}_{;\mu}^{;\mu} = -4fE_{;\mu}^{;\mu} - 2f_{;\mu}^{;\mu}E + \frac{1}{2}(n-2)f_{;\mu\nu}^{;\mu\nu} - 2f_{;\mu}E^{;\mu} \ , \tag{69}$$

$$\left.\frac{d}{d\epsilon}\right|_{\epsilon=0} \tilde{R}\tilde{E} = -4fRE + \frac{1}{2}(n-2)f_{;\mu}^{;\mu}R - 2(n-1)f_{;\mu}^{;\mu}E \ , \tag{70}$$

$$\left.\frac{d}{d\epsilon}\right|_{\epsilon=0} \tilde{E}^2 = -4fE^2 + (n-2)f_{;\mu}^{;\mu}E \ , \tag{71}$$

$$\left.\frac{d}{d\epsilon}\right|_{\epsilon=0} \tilde{R}_{;\mu}^{;\mu} = -4fR_{;\mu}^{;\mu} - 2f_{;\mu}^{;\mu}R - 2(n-1)f_{;\mu\nu}^{;\mu\nu} - 2f_{;\mu}R^{;\mu} \ , \tag{72}$$

$$\left.\frac{d}{d\epsilon}\right|_{\epsilon=0} \tilde{R}^2 = -4fR^2 - 4(n-1)f_{;\mu}^{;\mu}R \ , \tag{73}$$

$$\left.\frac{d}{d\epsilon}\right|_{\epsilon=0} \tilde{R}_{\mu\nu}\tilde{R}^{\mu\nu} = -4fR_{\mu\nu}R^{\mu\nu} - 2f_{;\mu}^{;\mu}R - 2(n-2)f_{;\mu\nu}R^{\mu\nu} \ , \tag{74}$$

$$\left.\frac{d}{d\epsilon}\right|_{\epsilon=0} \tilde{R}_{\mu\nu\rho\sigma}\tilde{R}^{\mu\nu\rho\sigma} = -4fR_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} - 8f_{;\mu\nu}R^{\mu\nu} \ , \tag{75}$$

$$\left.\frac{d}{d\epsilon}\right|_{\epsilon=0} \tilde{\Omega}^2 = -4f\Omega^2 \ . \tag{76}$$

- With this relations we can now use (58) to compute the coefficients. Applying it to $n=4$ we have

$$\left.\frac{d}{d\epsilon}\right|_{\epsilon=0} a_2(\exp^{-2\epsilon f} F, \exp^{-2\epsilon f} D) = 0 \tag{77}$$

and using (67) and (68)

$$\begin{aligned} \left.\frac{d}{d\epsilon}\right|_{\epsilon=0} a_2(\exp^{-2\epsilon f} F, \exp^{-2\epsilon f} D) &= (4\pi)^{-n/2}\frac{1}{6}\int_{\mathcal{M}} d^n x\sqrt{g}\ \mathrm{Tr}\left\{F(\alpha_1(-2fE\right. \\ &+ \left.\frac{1}{2}(n-2)f_{;\mu}^{;\mu}) + \alpha_2(-2fR - 2(n-1)f_{;\mu}^{;\mu})\right\} \ . \end{aligned} \tag{78}$$

Each coefficient in front of the different invariants must be zero, so that collecting the factors that go with $f_{;\mu}^{;\mu}$ we get

$$\boxed{\alpha_2 = 1} \longrightarrow \boxed{\alpha_4 = 60} \ . \tag{79}$$

[3] In these formulas ; denotes the covariant derivation.

- Now let $\mathcal{M} = \mathcal{M}_1 \times \mathcal{M}_2$ and let $D = (-\Delta_1) + (-\Delta_2)$ where $\Delta_{1,2}$ are scalar Laplacians. Then using the expression for the coefficients for a product manifold (49) we have

$$a_4(1, -\Delta_1 - \Delta_2) = a_4(1, \Delta_1)a_0(1, \Delta_2) + a_2(1, \Delta_1)a_2(1, \Delta_2) + a_0(1, \Delta_1)a_4(1, \Delta_2) \ . \tag{80}$$

In this case $E = 0$ and $\Omega = 0$ (we just have the scalar Laplacian). We want to collect the terms with $R_1 R_2$ to equate them and obtain the value of some coefficients. In the total a_4 we will have the term $\alpha_7(R_1 + R_2)^2$, whereas in the product of the coefficients $a_2 a_2$ we have $\alpha_2^2 R_1 R_2$, so that

$$\frac{2\alpha_7\ R_1 R_2}{360} = \frac{\alpha_2^2\ R_1 R_2}{36} \longrightarrow \boxed{\alpha_7 = 5} \ . \tag{81}$$

- Next, we can apply (58) to $n = 6$ and collect the coefficients multiplying each invariant imposing their cancellation

$$\begin{aligned} 0 \ &= \ \mathrm{Tr}_{\mathcal{V}} \int_{\mathcal{M}} d^n x \sqrt{g}\ \{F((-2\alpha_3 - 10\alpha_4 + 4\alpha_5) f_{;\mu}^{;\mu} + (2\alpha_3 - 10\alpha_6) f_{;\mu\nu}^{;\mu\nu} \\ &+ \ (2\alpha_4 - 2\alpha_6 - 20\alpha_7 - 2\alpha_8) f_{;\mu}^{;\mu} R + (-8\alpha_8 - 8\alpha_9) f_{\mu\nu} R^{\mu\nu})\} \ . \end{aligned} \tag{82}$$

Hence, we obtain the following system of equations

$$2\alpha_3 = 4\alpha_5 - 10a_4 \ ,$$

$$2\alpha_3 = 10\alpha_6 \ ,$$

$$\alpha_8 = \alpha_4 - \alpha_6 - 10\alpha_7 \ , \tag{83}$$

$$\alpha_9 = -\alpha_8 \ . \tag{84}$$

whose solution, taking into account the value of the computed coefficients, is

$$\boxed{\alpha_3 = 60} \quad \boxed{\alpha_6 = 12} \quad \boxed{\alpha_8 = -2} \quad \boxed{\alpha_9 = 2} \ . \tag{85}$$

- The computation of α_{10} is more tedious and it is left for (A.3) where we get $\boxed{\alpha_{10} = 30}$

Finally, we arrive at the known formulas for the first three heat kernel coefficients

$$a_0(f, D) \ = \ (4\pi)^{-n/2} \int_{\mathcal{M}} d^n x \sqrt{g}\ \mathrm{Tr}_{\mathcal{V}} \{f\} \ , \tag{86}$$

$$a_2(f, D) \ = \ (4\pi)^{-n/2} \frac{1}{6} \int_{\mathcal{M}} d^n x \sqrt{g}\ \mathrm{Tr}_{\mathcal{V}} \{f(6E + R)\} \ , \tag{87}$$

$$\begin{aligned} a_4(f, D) \ &= \ (4\pi)^{-n/2} \frac{1}{360} \int_{\mathcal{M}} d^n x \sqrt{g}\ \mathrm{Tr}_{\mathcal{V}} \{f(60 E_{;\mu}^{;\mu} + 60 RE + 180 E^2 + 12 R_{;\mu}^{;\mu} + \\ &+ \ 5R^2 + -2R_{\mu\nu}R^{\mu\nu} + 2R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} + 30\Omega_{\mu\nu}\Omega^{\mu\nu})\} \ . \end{aligned} \tag{88}$$

For our goal, we need to use (88) which is the coefficient providing the infinite part of the determinant of the operator to one-loop order.

4 Computation of the counterterms for the Einstein-Hilbert action in first order formalism

Let us analyze the first order Einstein-Hilbert action, in which the metric and the connection are independent and are expanded in a background field and a perturbation

$$\begin{aligned} g_{\mu\nu} &= \bar{g}_{\mu\nu} + \kappa h_{\mu\nu} \ , \\ \Gamma^{\mu}_{\nu\rho} &= \bar{\Gamma}^{\mu}_{\nu\rho} + A^{\mu}_{\nu\rho} \ , \end{aligned} \tag{89}$$

where indices are raised with the background metric, and the quantities computed with respect to this metric have a bar.

We also need the expansions for the inverse metric and the determinant, which are

$$g^{\mu\nu} = \bar{g}^{\mu\nu} - \kappa h^{\mu\nu} + \kappa^2 h^{\mu}_{\lambda} h^{\lambda\nu} + \mathcal{O}(\kappa^3) \ , \tag{90}$$

$$\sqrt{g} = \sqrt{\bar{g}} \left\{ 1 + \kappa \frac{h}{2} + \kappa^2 \left(\frac{h^2}{8} - \frac{h_{\mu\nu} h^{\mu\nu}}{4} \right) \right\} + \mathcal{O}(\kappa^3) \ . \tag{91}$$

The expression (90) is straightforward and for (91) we use

$$\begin{aligned} \sqrt{\det(g)} &= \exp\left(\frac{1}{2}\mathrm{Tr}\ln g\right) = \sqrt{\bar{g}}\exp\left(\frac{1}{2}\mathrm{Tr}\ln(\delta^{\alpha}_{\nu} + \kappa h^{\alpha}_{\nu})\right) = \\ &= \sqrt{\bar{g}}\exp\left(\frac{1}{2}\mathrm{Tr}(\kappa h^{\alpha}_{\nu} - \frac{1}{2}\kappa^2 h^{\alpha}_{\beta} h^{\beta}_{\nu})\right) = \sqrt{\bar{g}}\exp\left(\kappa\frac{1}{2}h^{\alpha}_{\alpha} - \frac{\kappa^2}{4}h^{\alpha}_{\beta}h^{\beta}_{\alpha}\right) \simeq \\ &\simeq \sqrt{\bar{g}}\left(1 + \frac{\kappa}{2}h - \frac{\kappa^2}{4}h^{\alpha}_{\beta}h^{\beta}_{\alpha} + \frac{\kappa^2}{8}h^2\right) + \mathcal{O}(\kappa^3) \ , \end{aligned} \tag{92}$$

where we have used the Taylor expansion of $\ln(1+x)$ and e^x for small x. To take into account the one-loop effects it is enough to expand the action up to quadratic order in the perturbations so that

$$\begin{aligned} S_{\text{FOEH}} \simeq\ & -\bar{S}_0 - \frac{1}{2\kappa^2}\int d^n x\ \kappa h^{\alpha\beta}\sqrt{|\bar{g}|}\left\{\frac{1}{2}g_{\alpha\beta}\bar{R} - \bar{R}_{\alpha\beta}\right\} + \\ &+ \int d^n x\ \sqrt{|\bar{g}|}\bar{g}^{\mu\nu}\left\{\delta^{\alpha}_{\mu}\left(\delta^{\beta}_{\nu}\bar{\nabla}_{\lambda} - \delta^{\beta}_{\lambda}\bar{\nabla}_{\nu}\right)\right\}A^{\lambda}_{\alpha\beta} + \\ &- \frac{1}{2\kappa^2}\int d^n x\ \kappa^2 h^{\alpha\beta}\sqrt{|\bar{g}|}\left\{\left(\frac{1}{8}\bar{g}_{\alpha\beta}\bar{g}_{\gamma\epsilon} - \frac{1}{4}\bar{g}_{\alpha\gamma}\bar{g}_{\beta\epsilon}\right)\bar{R} - \frac{1}{2}\bar{g}_{\alpha\beta}\bar{R}_{\gamma\epsilon} + \bar{g}_{\alpha\gamma}\bar{R}_{\beta\epsilon}\right\}h^{\gamma\epsilon} + \\ &- \frac{1}{2\kappa^2}\int d^n x\ \kappa h^{\gamma\epsilon}\sqrt{|\bar{g}|}\left\{\left(\frac{1}{2}\bar{g}_{\gamma\epsilon}\bar{g}^{\mu\nu} - \delta^{\mu}_{\gamma}\delta^{\nu}_{\epsilon}\right)\delta^{\alpha}_{\mu}\left(\delta^{\beta}_{\nu}\bar{\nabla}_{\lambda} - \delta^{\beta}_{\lambda}\bar{\nabla}_{\nu}\right)\right\}A^{\lambda}_{\alpha\beta} + \\ &- \frac{1}{2\kappa^2}\int d^n x\ \sqrt{|\bar{g}|}\bar{g}^{\mu\nu}A^{\tau}_{\gamma\epsilon}\left\{\delta^{\epsilon}_{\tau}\delta^{\gamma}_{\lambda}\delta^{\alpha}_{\mu}\delta^{\beta}_{\nu} - \delta^{\beta}_{\tau}\delta^{\gamma}_{\lambda}\delta^{\alpha}_{\mu}\delta^{\epsilon}_{\nu}\right\}A^{\lambda}_{\alpha\beta} \end{aligned} \tag{93}$$

.

As we can saw in the introduction of the background has to obey the classical equations of motion so that

$$\left.\frac{\delta S}{\delta g_{\mu\nu}}\right|_{g_{\mu\nu}=\bar{g}_{\mu\nu}} = \left.\frac{\delta S}{\delta h_{\mu\nu}}\right|_{h_{\mu\nu}=0} = 0 \ , \tag{94}$$

and thus, we can cancel the linear terms by imposing the background equations of motion

$$\frac{\delta S}{\delta h^{\alpha\beta}}\Big|_{h^{\alpha\beta}=0} \Rightarrow \frac{1}{2}g_{\alpha\beta}\bar{R} - \bar{R}_{\alpha\beta} = 0 \ ,$$
$$\frac{\delta S}{\delta A^{\lambda}_{\alpha\beta}}\Big|_{A^{\lambda}_{\alpha\beta}=0} = 0 \Rightarrow \int d^n x \ \sqrt{|\bar{g}|}\bar{g}^{\mu\nu}\left\{\delta^{\alpha}_{\mu}\left(\delta^{\beta}_{\nu}\bar{\nabla}_{\lambda} - \delta^{\beta}_{\lambda}\bar{\nabla}_{\nu}\right)\right\}A^{\lambda}_{\alpha\beta} = 0 \Rightarrow \bar{\nabla}_{\lambda}(\bar{g}^{\mu\nu}) = 0 \ , \tag{95}$$

so that we see that we recover Einstein's equations and the fact that the background connection is the Levi-Civitta connection.

After background field expansion then we can write the action as

$$\begin{aligned} S_{\text{FOEH}} &= -\bar{S}_0 - \int d^n x \ \sqrt{|g|} \ \left\{\frac{1}{2}h^{\alpha\beta}M_{\alpha\beta\gamma\epsilon}h^{\gamma\epsilon} + h^{\gamma\epsilon}\vec{N}_{\gamma\epsilon}{}^{\alpha\beta}{}_{\lambda}A^{\lambda}_{\alpha\beta} + \right. \\ &+ \left. \frac{1}{2}A^{\tau}_{\gamma\epsilon}K^{\gamma\epsilon}_{\tau}{}^{\alpha\beta}_{\lambda}A^{\lambda}_{\alpha\beta}\right\} \ , \end{aligned} \tag{96}$$

where the operators mediating the interaction between the relevant fields are

$$\begin{aligned} M_{\alpha\beta\gamma\epsilon} &= \left\{\frac{1}{8}\left(\bar{g}_{\alpha\beta}\bar{g}_{\gamma\epsilon} - \bar{g}_{\alpha\gamma}\bar{g}_{\beta\epsilon} - \bar{g}_{\alpha\epsilon}\bar{g}_{\beta\gamma}\right)\bar{R} - \right. \\ &- \left. \frac{1}{4}\left(\bar{g}_{\alpha\beta}\bar{R}_{\gamma\epsilon} + \bar{g}_{\gamma\epsilon}\bar{R}_{\alpha\beta} - \bar{g}_{\alpha\gamma}\bar{R}_{\beta\epsilon} - \bar{g}_{\alpha\epsilon}\bar{R}_{\beta\gamma} - \bar{g}_{\beta\gamma}\bar{R}_{\alpha\epsilon} - \bar{g}_{\beta\epsilon}\bar{R}_{\alpha\gamma}\right)\right\} \ , \\ N_{\gamma\epsilon}{}^{\alpha\beta}{}_{\lambda} &= \frac{1}{2\kappa}\left\{\frac{1}{2}\left(\bar{g}_{\gamma\epsilon}\bar{g}^{\alpha\beta} - \delta^{\alpha}_{\gamma}\delta^{\beta}_{\epsilon} - \delta^{\alpha}_{\epsilon}\delta^{\beta}_{\gamma}\right)\bar{\nabla}_{\lambda} - \right. \\ &- \left. \frac{1}{4}\left(\bar{g}_{\gamma\epsilon}\delta^{\beta}_{\lambda}\bar{\nabla}^{\alpha} - \delta^{\alpha}_{\gamma}\delta^{\beta}_{\lambda}\bar{\nabla}_{\epsilon} - \delta^{\alpha}_{\epsilon}\delta^{\beta}_{\lambda}\bar{\nabla}_{\gamma} + \bar{g}_{\gamma\epsilon}\delta^{\alpha}_{\lambda}\bar{\nabla}^{\beta} - \delta^{\beta}_{\gamma}\delta^{\alpha}_{\lambda}\bar{\nabla}_{\epsilon} - \delta^{\beta}_{\epsilon}\delta^{\alpha}_{\lambda}\bar{\nabla}_{\gamma}\right)\right\} \ , \\ K^{\gamma\epsilon}_{\tau}{}^{\alpha\beta}_{\lambda} &= \frac{1}{\kappa^2}\left\{\frac{1}{4}[\delta^{\epsilon}_{\tau}\delta^{\gamma}_{\lambda}\bar{g}^{\alpha\beta} + \delta^{\gamma}_{\tau}\delta^{\epsilon}_{\lambda}\bar{g}^{\alpha\beta} - \delta^{\beta}_{\tau}\delta^{\gamma}_{\lambda}\bar{g}^{\alpha\epsilon} - \delta^{\beta}_{\tau}\delta^{\epsilon}_{\lambda}\bar{g}^{\alpha\gamma} - \delta^{\alpha}_{\tau}\delta^{\epsilon}_{\lambda}\bar{g}^{\beta\gamma} - \delta^{\alpha}_{\tau}\delta^{\gamma}_{\lambda}\bar{g}^{\beta\epsilon}\right. \\ &+ \left. \delta^{\beta}_{\lambda}\delta^{\alpha}_{\tau}\bar{g}^{\gamma\epsilon} + \delta^{\alpha}_{\lambda}\delta^{\beta}_{\tau}\bar{g}^{\gamma\epsilon}\right\} \ . \end{aligned} \tag{97}$$

Note that the arrow on $\vec{N}$ means that the operator acts on the right side and this is important because if we want it to act on the left there is a minus due to integration by parts.

We are interested in computing the path integral

$$\begin{aligned} e^{iW} &= \int \mathcal{D}h\mathcal{D}A \ \exp\left\{-i\int d^n x \ \sqrt{|g|} \ (\bar{S}_0 + \frac{1}{2}h^{\alpha\beta}M_{\alpha\beta\gamma\epsilon}h^{\gamma\epsilon} + \frac{1}{2}\left(h^{\gamma\epsilon}\vec{N}_{\gamma\epsilon}{}^{\alpha\beta}{}_{\lambda}A^{\lambda}_{\alpha\beta} - A^{\lambda}_{\alpha\beta}\vec{N}^{\alpha\beta}{}_{\lambda}{}_{\gamma\epsilon}h^{\gamma\epsilon}\right)\right. \\ &+ \left. \frac{1}{2}A^{\tau}_{\gamma\epsilon}K^{\gamma\epsilon}_{\tau}{}^{\alpha\beta}_{\lambda}A^{\lambda}_{\alpha\beta})\right\} \ . \end{aligned} \tag{98}$$

First we are performing the integral in A completing squares and performing the general gaussian integral. For that, we can write the terms in the exponential as (we do not write the indices for simplicity)

$$\frac{1}{2}hMh + h\vec{N}A + \frac{1}{2}AKA = \frac{1}{2}hMh + \frac{1}{2}[h\vec{N} + AK]K^{-1}[-\vec{N}h + KA] + \frac{1}{2}h\vec{N}K^{-1}\vec{N}h \tag{99}$$

We know that the measure $\mathcal{D}A$ of the path integral is translational invariant so that we can compute the integral in A like a usual gaussian integral which yields $(\det K)^{-1/2}$. Nevertheless, this determinant is finite and we can always ignore all the constant pieces taking them as contributions to the $\bar{S}_0$. Let us note that we are allowed to do this because the operator K only contains background metrics and thus is finite and we can treat it as a constant. So that after this we can write

$$\bar{S}_0' = \bar{S}_0 - \frac{1}{2}\ln\det K \tag{100}$$

and finally the integration over A yields

$$\begin{aligned} e^{iW[\bar{g}_{\mu\nu},\bar{\Gamma}^\lambda_{\rho\sigma}]} &= \int \mathcal{D}h \, \exp\left\{-\frac{i}{2}\int d^nx\,\sqrt{|g|}\,h^{\mu\nu}\left(\bar{S}_0' + M_{\mu\nu\rho\sigma} + \frac{1}{2}N_{\mu\nu}{}^{\alpha\beta}{}_\lambda (K^{-1})_{\alpha\beta}^{\lambda}{}_{\gamma\epsilon}^{\tau} N^{\gamma\epsilon}_{\tau}{}_{\rho\sigma} + \right.\right. \\ &+ \left.\left. \frac{1}{2}N_{\rho\sigma}{}^{\alpha\beta}{}_\lambda (K^{-1})_{\alpha\beta}^{\lambda}{}_{\gamma\epsilon}^{\tau} N^{\gamma\epsilon}_{\tau}{}_{\mu\nu}\right)h^{\rho\sigma}\right\} . \end{aligned} \tag{101}$$

The last step is the integration over h that gives another determinant. In this case, this determinant has an infinite part that will give us the counterterm we are looking for. Therefore, we have

$$W[\bar{g}_{\mu\nu},\bar{\Gamma}^\lambda_{\rho\sigma}] \equiv \bar{S}_0'[\bar{g}_{\mu\nu},\bar{\Gamma}^\lambda_{\rho\sigma}] - \frac{1}{2}\log\,\det\,\Delta[\bar{g}_{\mu\nu},\bar{\Gamma}^\lambda_{\rho\sigma}] \,, \tag{102}$$

and as we said the counterterm will be given by the infinite part of the determinant

$$\Delta S[\bar{g}_{\mu\nu},\bar{\Gamma}^\lambda_{\rho\sigma}] = -\frac{1}{2}\log\,\det\,\Delta[\bar{g}_{\mu\nu},\bar{\Gamma}^\lambda_{\rho\sigma}] \,. \tag{103}$$

We can also write that the action in the first order formalism and to one loop as

$$\begin{aligned} S_{\text{FOEH}} &= -\bar{S}_0' + \frac{1}{2}\int d^nx\,\sqrt{|g|}\,\frac{1}{2}\,h^{\mu\nu}\Delta_{\mu\nu\rho\sigma}h^{\rho\sigma} = \\ &= -\bar{S}_0' + \frac{1}{2}\int d^nx\,\sqrt{|g|}\,\frac{1}{2}\,h^{\mu\nu}\left(-D_{\mu\nu\rho\sigma} - 2M_{\mu\nu\rho\sigma}\right)h^{\rho\sigma} \,, \end{aligned} \tag{104}$$

where

$$D_{\mu\nu\rho\sigma} = N_{\mu\nu}{}^{\alpha\beta}{}_\lambda (K^{-1})_{\alpha\beta}^{\lambda}{}_{\gamma\epsilon}^{\tau} N^{\gamma\epsilon}_{\tau}{}_{\rho\sigma} + N_{\rho\sigma}{}^{\alpha\beta}{}_\lambda (K^{-1})_{\alpha\beta}^{\lambda}{}_{\gamma\epsilon}^{\tau} N^{\gamma\epsilon}_{\tau}{}_{\mu\nu} \,. \tag{105}$$

For this calculation we need the inverse of $K^{\gamma\epsilon}_{\tau}{}^{\alpha\beta}_{\lambda}$, which without gauge fixing is[4]

$$\begin{aligned} (K^{-1})_{\alpha\beta}^{\lambda}{}_{\gamma\epsilon}^{\tau} &= \kappa^2\left\{\frac{1}{n-2}\left\{\delta_\gamma{}^\tau\delta_\epsilon{}^\lambda\bar{g}_{\alpha\beta} + \delta_\gamma{}^\lambda\delta_\epsilon{}^\tau\bar{g}_{\alpha\beta} + \delta_\alpha{}^\tau\delta_\beta{}^\lambda\bar{g}_{\gamma\epsilon} + \delta_\alpha{}^\lambda\delta_\beta{}^\tau\bar{g}_{\gamma\epsilon}\right\}\right. \\ &- \frac{1}{2}\delta_\beta{}^\tau\delta_\epsilon{}^\lambda\bar{g}_{\alpha\gamma} - \frac{1}{2}\delta_\alpha{}^\tau\delta_\epsilon{}^\lambda\bar{g}_{\beta\gamma} - \frac{1}{2}\delta_\beta{}^\tau\delta_\gamma{}^\lambda\bar{g}_{\alpha\epsilon} - \frac{1}{2}\delta_\alpha{}^\tau\delta_\gamma{}^\lambda\bar{g}_{\beta\epsilon} - \\ &- \frac{1}{n^2-3n+2}\left\{\delta_\beta{}^\lambda\delta_\epsilon{}^\tau\bar{g}_{\alpha\gamma} + \delta_\alpha{}^\lambda\delta_\epsilon{}^\tau\bar{g}_{\beta\gamma} + \delta_\beta{}^\lambda\delta_\gamma{}^\tau\bar{g}_{\alpha\epsilon} + \delta_\alpha{}^\lambda\delta_\gamma{}^\tau\bar{g}_{\beta\epsilon}\right\} + \\ &\left. +\frac{1}{2}\bar{g}_{\alpha\epsilon}\bar{g}_{\beta\gamma}\bar{g}^{\lambda\tau} + \frac{1}{2}\bar{g}_{\alpha\gamma}\bar{g}_{\beta\epsilon}\bar{g}^{\lambda\tau} - \frac{1}{n-2}\bar{g}_{\alpha\beta}\bar{g}_{\gamma\epsilon}\bar{g}^{\lambda\tau}\right\} . \end{aligned} \tag{106}$$

[4]The inverse of the operator K is computed using the *xAct* package in *Mathematica*.

Before continuing, one may ask why is it the case that we can invert the operator K without fixing a gauge and thus not taking care the zero mode. To understand this, we note that the gauge invariance is preserved for the whole action (96) and hence this implies (omitting the indices)

$$\delta S_{\text{FOEH}} = hM\ \delta h + \delta h\ NA + hN\ \delta A + AK\ \delta A = 0\ . \tag{107}$$

We know that h and A are independent fields and therefore we obtain two equations

$$M\ \delta h + N\ \delta A \ = \ 0\ , \tag{108}$$

$$\delta h N + K\ \delta A \ = \ 0\ . \tag{109}$$

As we can see in (109), the gauge invariance of the whole action does not imply that K has a zero mode. It could be the case that both N and K have a zero mode, but in this case it can be checked that when doing background gauge transformations, it is the sum of both terms what cancels, and hence, we can invert K without introducing a gauge fixing term.

Finally, multiplying the obtained result with the operator N we get

$$\begin{aligned} D_{\mu\nu\rho\sigma} \ =\ & \frac{1}{4}(\bar{g}_{\mu\rho}\bar{g}_{\nu\sigma} + \bar{g}_{\mu\sigma}\bar{g}_{\nu\rho} - 2\bar{g}_{\mu\nu}\bar{g}_{\rho\sigma})\Box + \frac{1}{2}(\bar{g}_{\mu\nu}\bar{\nabla}_\rho\bar{\nabla}_\sigma + \bar{g}_{\rho\sigma}\bar{\nabla}_\mu\bar{\nabla}_\nu) \\ -\ & \frac{1}{8}(\bar{g}_{\mu\rho}\bar{\nabla}_\nu\bar{\nabla}_\sigma + \bar{g}_{\mu\sigma}\bar{\nabla}_\nu\bar{\nabla}_\rho + \bar{g}_{\nu\rho}\bar{\nabla}_\mu\bar{\nabla}_\sigma + \bar{g}_{\nu\sigma}\bar{\nabla}_\mu\bar{\nabla}_\rho) \\ -\ & \frac{1}{8}(\bar{g}_{\mu\rho}\bar{\nabla}_\sigma\bar{\nabla}_\nu + \bar{g}_{\mu\sigma}\bar{\nabla}_\rho\bar{\nabla}_\nu + \bar{g}_{\nu\rho}\bar{\nabla}_\sigma\bar{\nabla}_\mu + \bar{g}_{\nu\sigma}\bar{\nabla}_\rho\bar{\nabla}_\mu) - \\ +\ & \frac{1}{8}\left(\bar{g}_{\mu\rho}\bar{R}_{\nu\sigma} + \bar{g}_{\mu\sigma}\bar{R}_{\nu\rho} + \bar{g}_{\nu\rho}\bar{R}_{\mu\sigma} + \bar{g}_{\nu\sigma}\bar{R}_{\mu\rho}\right) - \frac{1}{4}\left(\bar{R}_{\mu\rho\nu\sigma} + \bar{R}_{\nu\rho\mu\sigma}\right)\ , \end{aligned} \tag{110}$$

so that we have

$$S_{\text{FOEH}} \ = \ -\bar{S}_0' + \frac{1}{2}\int d^n x\ \sqrt{|g|}\ \frac{1}{2}\ h^{\mu\nu}\left(-D_{\mu\nu\rho\sigma} - 2M_{\mu\nu\rho\sigma}\right)h^{\rho\sigma}\ , \tag{111}$$

where

$$\begin{aligned} -D_{\mu\nu\rho\sigma} - 2M_{\mu\nu\rho\sigma} \ =\ & \frac{1}{4}(\bar{g}_{\mu\rho}\bar{g}_{\nu\sigma} + \bar{g}_{\mu\sigma}\bar{g}_{\nu\rho} - \bar{g}_{\mu\nu}\bar{g}_{\rho\sigma})\bar{R} + \frac{1}{2}\left(\bar{g}_{\mu\nu}\bar{R}_{\rho\sigma} + \bar{g}_{\rho\sigma}\bar{R}_{\mu\nu}\right) - \\ -\ & \frac{1}{2}\left(\bar{g}_{\mu\rho}\bar{R}_{\nu\sigma} + \bar{g}_{\mu\sigma}\bar{R}_{\nu\rho} + \bar{g}_{\nu\rho}\bar{R}_{\mu\sigma} + \bar{g}_{\nu\sigma}\bar{R}_{\mu\rho}\right) \\ -\ & \frac{1}{4}(\bar{g}_{\mu\rho}\bar{g}_{\nu\sigma} + \bar{g}_{\mu\sigma}\bar{g}_{\nu\rho} - 2\bar{g}_{\mu\nu}\bar{g}_{\rho\sigma})\Box - \frac{1}{2}(\bar{g}_{\mu\nu}\bar{\nabla}_\rho\bar{\nabla}_\sigma - \bar{g}_{\rho\sigma}\bar{\nabla}_\mu\bar{\nabla}_\nu) \\ +\ & \frac{1}{8}(\bar{g}_{\mu\rho}\bar{\nabla}_\nu\bar{\nabla}_\sigma + \bar{g}_{\mu\sigma}\bar{\nabla}_\nu\bar{\nabla}_\rho + \bar{g}_{\nu\rho}\bar{\nabla}_\mu\bar{\nabla}_\sigma + \bar{g}_{\nu\sigma}\bar{\nabla}_\mu\bar{\nabla}_\rho) \\ +\ & \frac{1}{8}(\bar{g}_{\mu\rho}\bar{\nabla}_\sigma\bar{\nabla}_\nu + \bar{g}_{\mu\sigma}\bar{\nabla}_\rho\bar{\nabla}_\nu + \bar{g}_{\nu\rho}\bar{\nabla}_\sigma\bar{\nabla}_\mu + \bar{g}_{\nu\sigma}\bar{\nabla}_\rho\bar{\nabla}_\mu) \end{aligned} \tag{112}$$

Moreover, as we saw in the introduction to the background field method, we want to gauge fix the quantum gauge transformations but leave the background gauge transformations untouched. We can use this freedom to fix the gauge for the quantum fields in a way that simplifies the computation. In order to do that, we will introduce the following gauge-fixing term

$$S_{\text{gf}} = \frac{1}{2} \int d^n x \, \sqrt{\bar{g}} \, \frac{1}{2\xi} \, \bar{g}_{\mu\nu} \chi^\mu \chi^\nu \ , \tag{113}$$

where the function characterizing the harmonic gauge is

$$\chi^\nu = \bar{\nabla}_\mu h^{\mu\nu} - \frac{1}{2} \bar{\nabla}^\nu h \ . \tag{114}$$

Introducing that function in (113) and integrating by parts, we get an operator mediating the interaction between two quantum fields h, which we can then add to (112) to cancel some of its terms

$$\begin{aligned}
S_{\text{gf}} &= \frac{1}{2} \int d^n x \, \sqrt{\bar{g}} \, \frac{1}{2\xi} \, \bar{g}_{\mu\nu} \left(\bar{\nabla}_\alpha h^{\alpha\mu} - \frac{1}{2} \bar{\nabla}^\mu h \right) \left(\bar{\nabla}_\beta h^{\beta\nu} - \frac{1}{2} \bar{\nabla}^\nu h \right) = \\
&= \frac{1}{2} \int d^n x \, \sqrt{\bar{g}} \, \frac{1}{2\xi} \, h^{\mu\nu} \Big(-\frac{1}{8} \left(\bar{g}_{\mu\rho} \bar{\nabla}_\nu \bar{\nabla}_\sigma + \bar{g}_{\nu\rho} \bar{\nabla}_\mu \bar{\nabla}_\sigma + \bar{g}_{\mu\sigma} \bar{\nabla}_\nu \bar{\nabla}_\rho + \bar{g}_{\nu\sigma} \bar{\nabla}_\mu \bar{\nabla}_\rho \right) + \\
&+ \frac{1}{4} \bar{g}_{\rho\sigma} \bar{\nabla}_\nu \bar{\nabla}_\mu + \frac{1}{4} \bar{g}_{\mu\nu} \bar{\nabla}_\rho \bar{\nabla}_\sigma - \frac{1}{8} \bar{g}_{\mu\nu} \bar{g}_{\rho\sigma} \Box \Big) h^{\rho\sigma} + \\
&+ \frac{1}{2} \int d^n x \, \sqrt{\bar{g}} \, \frac{1}{2\xi} \, h^{\mu\nu} \Big(-\frac{1}{8} \left(\bar{g}_{\mu\rho} \bar{\nabla}_\sigma \bar{\nabla}_\nu + \bar{g}_{\nu\rho} \bar{\nabla}_\sigma \bar{\nabla}_\mu + \bar{g}_{\mu\sigma} \bar{\nabla}_\rho \bar{\nabla}_\nu + \bar{g}_{\nu\sigma} \bar{\nabla}_\rho \bar{\nabla}_\mu \right) + \\
&+ \frac{1}{4} \bar{g}_{\rho\sigma} \bar{\nabla}_\mu \bar{\nabla}_\nu + \frac{1}{4} \bar{g}_{\mu\nu} \bar{\nabla}_\sigma \bar{\nabla}_\rho - \frac{1}{8} \bar{g}_{\mu\nu} \bar{g}_{\rho\sigma} \Box \Big) h^{\rho\sigma} + \\
&+ \frac{1}{2} \int d^n x \, \sqrt{\bar{g}} \, \frac{1}{2\xi} \, h^{\mu\nu} \Big(+ \frac{1}{4} \left(\bar{g}_{\mu\rho} \bar{R}_{\nu\sigma} + \bar{g}_{\mu\sigma} \bar{R}_{\nu\rho} + \bar{g}_{\nu\rho} \bar{R}_{\mu\sigma} + \bar{g}_{\nu\sigma} \bar{R}_{\mu\rho} \right) \\
&- \frac{1}{2} \left(\bar{R}_{\mu\rho\nu\sigma} + \bar{R}_{\nu\rho\mu\sigma} \right) \Big) h^{\rho\sigma} \ ,
\end{aligned} \tag{115}$$

where we use that the commutator of two covariant derivatives applied to $h_{\alpha\beta}$ which has the form

$$[\bar{\nabla}_\mu, \bar{\nabla}_\nu] h^{\alpha\beta} = h^{\beta\lambda} \bar{R}^\alpha{}_{\lambda\mu\nu} + h^{\alpha\lambda} \bar{R}^\beta{}_{\lambda\mu\nu} \tag{116}$$

, i.e.

$$\bar{g}_{\mu\rho} \bar{\nabla}_\nu \bar{\nabla}_\sigma = \bar{g}_{\mu\rho} \bar{\nabla}_\sigma \bar{\nabla}_\nu + \bar{R}_{\mu\rho\nu\sigma} - \bar{g}_{\mu\rho} \bar{R}_{\sigma\nu} \tag{117}$$

and with $\xi = 1$ we get

$$S_{\text{FOEH}} + S_{gf} = -\bar{S}_0' + \frac{1}{2} \int d^n x \, \sqrt{|g|} \, \frac{1}{2} \, h^{\mu\nu} \Delta_{\mu\nu\rho\sigma} h^{\rho\sigma} \ . \tag{118}$$

Finally, the operator whose infinite part gives us the counterterm takes the form

$$\Delta_{\mu\nu\rho\sigma} = C_{\mu\nu\rho\sigma} \Box + Y_{\mu\nu\rho\sigma} \ , \tag{119}$$

with

$$
\begin{aligned}
C_{\mu\nu\rho\sigma} &= -\frac{1}{4}(\bar{g}_{\mu\rho}\bar{g}_{\nu\sigma}+\bar{g}_{\mu\sigma}\bar{g}_{\nu\rho}-\bar{g}_{\mu\nu}\bar{g}_{\rho\sigma}) \\
C^{\mu\nu\rho\sigma} &= -\left(\bar{g}^{\mu\rho}\bar{g}^{\nu\sigma}+\bar{g}^{\mu\sigma}\bar{g}^{\nu\rho}-\frac{2}{n-2}\bar{g}^{\mu\nu}\bar{g}^{\rho\sigma}\right) \\
Y_{\mu\nu\rho\sigma} &= \frac{1}{4}(\bar{g}_{\mu\rho}\bar{g}_{\nu\sigma}+\bar{g}_{\mu\sigma}\bar{g}_{\nu\rho}-\bar{g}_{\mu\nu}\bar{g}_{\rho\sigma})\bar{R}+\frac{1}{2}\left(\bar{g}_{\rho\sigma}\bar{R}_{\mu\nu}+\bar{g}_{\mu\nu}\bar{R}_{\rho\sigma}\right) \\
&- \frac{1}{4}\left(\bar{g}_{\nu\sigma}\bar{R}_{\mu\rho}+\bar{g}_{\mu\sigma}\bar{R}_{\nu\rho}+\bar{g}_{\nu\rho}\bar{R}_{\mu\sigma}+\bar{g}_{\mu\rho}\bar{R}_{\nu\sigma}\right)-\frac{1}{2}\left(\bar{R}_{\mu\alpha\nu\beta}+\bar{R}_{\nu\alpha\mu\beta}\right) \quad (120)
\end{aligned}
$$

At this point, we can use the formula for the short time heat kernel coefficients (88). We first have to identify the piece multiplying the second derivative terms and the operator remaining. In this case the formula reads[5]

$$
\mathrm{tr}\, a_2\,(x,x) = \mathrm{tr}\left\{\frac{1}{360}\left(2\bar{R}_{\mu\nu\rho\sigma}\bar{R}^{\mu\nu\rho\sigma}-2\bar{R}_{\mu\nu}\bar{R}^{\mu\nu}+5\bar{R}^2\right)I+\frac{1}{2}Y^2+\frac{1}{6}\bar{R}Y+\frac{1}{12}W_{\mu\nu}W^{\mu\nu}\right\}\ . \quad (121)
$$

We do not take into account the terms with $\Box E$ and $\Box R$ because they vanish as surface terms when we perform the integral that gives the counterterms [6] (let us recall that the counterterms are given by an integral).

The field strength is defined through

$$
[\bar{\nabla}_\mu,\bar{\nabla}_\nu]h^{\alpha\beta} = W^{\alpha\beta}_{\rho\sigma\mu\nu}h^{\rho\sigma} \quad (122)
$$

and with our conventions we have

$$
[\bar{\nabla}_\mu,\bar{\nabla}_\nu]h^{\alpha\beta} = h^{\beta\lambda}\bar{R}^{\alpha}{}_{\lambda\mu\nu}+h^{\alpha\lambda}\bar{R}^{\beta}{}_{\lambda\mu\nu}\ , \quad (123)
$$

so that

$$
W^{\alpha\beta}_{\rho\sigma\mu\nu} = \delta^{\beta}_{\sigma}\bar{R}^{\alpha}{}_{\rho\mu\nu}+\delta^{\alpha}_{\sigma}\bar{R}^{\beta}{}_{\rho\mu\nu}\ . \quad (124)
$$

In order to find the explicit value for the counterterms we need to compute the traces of (121). To compute the trace of the identity we recall that in (119) the operator $C_{\mu\nu\rho\sigma}$ is the one multiplying the Laplacian it so that it plays the role of the metric and therefore the trace of the identity is computed as the contraction between this operator and its inverse. The other traces are taken by contracting the indices with the inverse of

[5] We were calling a_4 to the coefficient but if we consider just the even ones and start from zero we have a_2. Both names are used in the literature.

[6] The integral is not appearing here because we are taking about $a_k(x, D)$ (see (45)).

the operator acting like the metric. This yields

$$
\begin{aligned}
\text{tr } I &= C^{\mu\nu\alpha\beta}C_{\mu\nu\alpha\beta} = \frac{n(n+1)}{2} \ , \\
\text{tr } Y &= C^{\mu\nu\alpha\beta}Y_{\mu\nu\alpha\beta} = -\frac{n(n-1)}{2}\bar{R} \ , \\
\text{tr } Y^2 &= Y_{\mu\nu\alpha\beta}C^{\alpha\beta\gamma\epsilon}Y_{\gamma\epsilon\lambda\tau}C^{\lambda\tau\mu\nu} = 3\bar{R}_{\mu\nu\rho\sigma}\bar{R}^{\mu\nu\rho\sigma} + \frac{n^2-8n+4}{n-2}\bar{R}_{\mu\nu}\bar{R}^{\mu\nu} + \\
&+ \frac{n^3-5n^2+8n+4}{2(n-2)}\bar{R}^2 \ , \\
\text{tr } W_{\mu\nu}W^{\mu\nu} &= W^{\alpha\beta}_{\rho\sigma\mu\nu}C^{\rho\sigma\lambda\tau}W^{\gamma\epsilon\mu\nu}_{\lambda\tau}C_{\gamma\epsilon\alpha\beta} = -(n+2)\bar{R}_{\mu\nu\rho\sigma}\bar{R}^{\mu\nu\rho\sigma} \ .
\end{aligned} \tag{125}
$$

Using the expresion (121) for a_2 and introducing the expressions for the traces we arrive to

$$
\begin{aligned}
\text{tr}\, a_2(x,x) &= \frac{1}{360}\Big\{ [n(n+1) - 30(n+2) + 540]\,\bar{R}_{\mu\nu\rho\sigma}\bar{R}^{\mu\nu\rho\sigma} + \\
&+ \left[180\frac{n^2-8n+4}{n-2} - n(n+1)\right]\bar{R}_{\mu\nu}\bar{R}^{\mu\nu} + \\
&+ \left[180\frac{n^3-5n^2+8n+4}{2(n-2)} + \frac{5n(n+1)}{2} - 60\frac{n(n-1)}{2}\right]\bar{R}^2\Big\}
\end{aligned} \tag{126}
$$

and finally, in $n = 4$ dimensions, we obtain

$$
\text{tr}\, a_2(x,x)\big|_{n=4} = \frac{7}{6}\bar{R}_{\mu\nu}\bar{R}^{\mu\nu} + \frac{7}{12}\bar{R}^2 \ , \tag{127}
$$

where we use the well-known identity coming from the Gauss-Bonnet theorem

$$
\bar{R}_{\mu\nu\rho\sigma}\bar{R}^{\mu\nu\rho\sigma} - 4\bar{R}_{\mu\nu}\bar{R}^{\mu\nu} + \bar{R}^2 = \text{total derivative} \ , \tag{128}
$$

which is true in only for four-dimensions.

The contribution coming from ghost loops is also needed. We wanted the gauge fixing term to mantain the background invariance, under which the background $\bar{g}_{\mu\nu}$ transforms as a metric and the fluctuation $h_{\mu\nu}$ as a tensor, but we also wanted to break the quantum symmetry. We can do this with the explicit transformations

$$
\begin{aligned}
\delta\bar{g}_{\mu\nu} &= 0 \ , \\
\delta h_{\mu\nu} &= 2\bar{\nabla}_{(\mu}\xi_{\nu)} + \kappa\ \mathcal{L}_\xi h_{\mu\nu} \ .
\end{aligned} \tag{129}
$$

The ghost Lagrangian is obtained performing a variation on the function χ_ν characterizing the harmonic gauge

$$
\begin{aligned}
\delta\chi_\nu &= -\bar{\nabla}^\mu\delta h_{\mu\nu} + \frac{1}{2}\bar{g}^{\alpha\beta}\bar{\nabla}_\nu\delta h_{\alpha\beta} = \left(\bar{\nabla}^\mu\bar{\nabla}_\mu\xi_\nu - \left[\bar{\nabla}_\nu,\bar{\nabla}_\mu\right]\xi^\mu + \mathcal{O}(A)\right) = \\
&= -\left(\bar{\nabla}^2\bar{g}_{\mu\nu} + \bar{R}_{\mu\nu}\right)\xi^\mu + \mathcal{O}(A) \ .
\end{aligned} \tag{130}
$$

The terms leading to cubic operators in the fluctuations are irrelevant at one-loop. Ghosts are quantum fields and hence the terms including a quantum field A do not contribute to the ghost Lagrangian at this order. The Faddeev-Poppov determinant then reads

$$\det \frac{\delta \chi_\nu}{\delta \xi_\mu} = \det(-\bar{\nabla}^2 \bar{g}_{\mu\nu} - \bar{R}_{\mu\nu}) \ . \tag{131}$$

We take as the ghost action

$$S_{\rm gh} = \frac{1}{2} \int d^n x \ \sqrt{\bar{g}} \, \frac{1}{2} \, V^*_\mu \left(-\Box \bar{g}^{\mu\nu} - \bar{R}^{\mu\nu}\right) V_\nu \ . \tag{132}$$

The relevant ghostly traces for the computation can be computed as before, obtaining

$$\begin{aligned} {\rm tr}\, I &= C^{\mu\nu} C_{\mu\nu} = n \ , \\ {\rm tr}\, Y &= C^{\mu\nu} Y_{\mu\nu} = \bar{R} \ , \\ {\rm tr}\, Y^2 &= Y_{\mu\nu} C^{\nu\alpha} Y_{\alpha\beta} C^{\beta\mu} = \bar{R}_{\mu\nu} \bar{R}^{\mu\nu} \ , \\ {\rm tr}\, W_{\mu\nu} W^{\mu\nu} &= -\bar{R}_{\mu\nu\rho\sigma} \bar{R}^{\mu\nu\rho\sigma} \ , \end{aligned} \tag{133}$$

where by (132)

$$\begin{aligned} C_{\mu\nu} &= -\bar{g}_{\mu\nu} \ , \\ Y_{\mu\nu} &= -\bar{R}_{\mu\nu} \ . \end{aligned} \tag{134}$$

Hence, the heat kernel coefficient coming from the ghost loops is

$${\rm tr}\, a_2^{gh}(x,x) = \frac{1}{360} \left[(2n-30) \bar{R}_{\mu\nu\rho\sigma} \bar{R}^{\mu\nu\rho\sigma} - (2n-180) \bar{R}_{\mu\nu} \bar{R}^{\mu\nu} + (5n+60) \bar{R}^2 \right] \ . \tag{135}$$

This yields in $n=4$ dimensions

$$\left. {\rm tr}\, a_2^{gh}(x,x) \right|_{n=4} = \frac{7}{30} \bar{R}_{\mu\nu} \bar{R}^{\mu\nu} + \frac{17}{60} \bar{R}^2 \tag{136}$$

where we have again used (128). Adding the two pieces (127) and (136), the one-loop counterterm is obtained

$$\begin{aligned} \Delta S_{\rm FOEH} &= \frac{1}{16\pi^2} \frac{1}{n-4} \int d^n x \ \sqrt{|g|} \left({\rm tr}\, a_2(x,x) - 2 {\rm tr}\, a_2^{gh}(x,x) \right) = \\ &= \frac{1}{16\pi^2} \frac{1}{n-4} \int d^n x \ \sqrt{|g|} \left(\frac{7}{10} \bar{R}_{\mu\nu} \bar{R}^{\mu\nu} + \frac{1}{60} \bar{R}^2 \right) \ . \end{aligned} \tag{137}$$

The ghost coefficient carries a different sign because of its fermionic character and a factor of two because they are imaginary fields.

The result obtained by 't Hooft and Veltman [2] is

$$\Delta S_{\rm SOEH} = \frac{1}{16\pi^2} \frac{1}{n-4} \int d^n x \ \sqrt{|g|} \left(\frac{7}{10} \bar{R}_{\mu\nu} \bar{R}^{\mu\nu} + \frac{1}{60} \bar{R}^2 \right) \tag{138}$$

therefore we see that

$$\Delta S_{\text{FOEH}} = \Delta S_{\text{SOEH}} \tag{139}$$

To end with this section, we can see that when imposing the background equations of motion (95) the counterterm cancels so that we have

$$\Delta S_{\text{ FOEH}} = 0 \ . \tag{140}$$

As expected, pure gravity is finite at one-loop order.

5 Conclusions

In this work we have computed the one-loop counterterms for the Einstein-Hilbert action in first order formalism using the backgound field technique and the heat kernel method. The computations have been carried out with the same field parametrizations and the same gauge fixing that 't Hooft and Veltmann used in their second order calculation. The result obtained for the counterterms matches exactly their result in [2]. Again, we recover the result that pure gravity is finite on-shell at one-loop.

First order and second order formalisms are known to be equivalent at tree level, as found by Palatini in 1919. Nevertheless, there is no reason to think that this equivalence must hold at the quantum level. Some research has been done on the on-shell equivalence of both formalisms [15, 16]. Still, there is no proof that the counterterms must coincide off-shell in first order and second order computations. From our result we can conclude that when using the same parametrizations and gauge fixing, the first and second order formalisms are equivalent for the Einstein Hilbert action to one-loop order, even off-shell.

Moreover, we have seen that the heat kernel technique has turned out to be very useful to shorten the computation, as we have been able to extract the counterterm by computing just one coefficient of the small proper time expansion of the heat kernel, with no need to focus on the different divergent diagrams of the theory.

To conclude, this is not the end of the road. Future work is going on regarding the study of quadratic theories in first order formalism. In this theories the extraction of the one-loop counterterms can only be handled with the help of heat kernel type techniques, so that this work is the first step in further work on this type of computations.

A Heat Kernel coefficients' computation

A.1 Variational equations for the heat kernel

In this appendix we derive the variational equations (56), (57) and (58). Starting with (56) we can write

$$\frac{d}{d\epsilon}\Big|_{\epsilon=0} \mathrm{Tr}(\exp(-e^{-2\epsilon f}\ tD)) = \mathrm{Tr}(2ftD\ e^{(-tD)}) = -2t\ \frac{d}{dt}\mathrm{Tr}(f\ e^{(-tD)}) \tag{141}$$

and taking the trace in both sides using (44) we get

$$\begin{aligned} \frac{d}{d\epsilon}\Big|_{\epsilon=0} \sum_{k\geq 0} t^{(k-n)/2} a_k(1, e^{-2\epsilon f D}) &= -2t\ (k-n)/2 \sum_{k\geq 0} t^{(k-n)/2-1} a_k(f, D) = \\ &= (n-k)\ \sum_{k\geq 0} t^{(k-n)/2} a_k(f, D)\ , \end{aligned} \tag{142}$$

so that for each k we have

$$\boxed{\frac{d}{d\epsilon}\Big|_{\epsilon=0} a_k(1, e^{-2\epsilon f} D) = (n-k) a_k(f, D)}\ . \tag{143}$$

For (57) we can write

$$\frac{d}{d\epsilon}\Big|_{\epsilon=0} \mathrm{Tr}(e^{-t(D-\epsilon F)}) = \mathrm{Tr}(tF\ e^{(-tD)}) \tag{144}$$

and again taking the trace in both sides we get

$$\begin{aligned} \frac{d}{d\epsilon}\Big|_{\epsilon=0} \sum_{k\geq 0} t^{(k-n)/2} a_k(1, D-\epsilon F) &= \sum_{k\geq 0} t^{(k-n)/2+1} a_k(F, D) = \\ &= \sum_{k'\geq 0} t^{(k'-n)/2} a_{k'-2}(F, D)\ , \end{aligned} \tag{145}$$

so that we can call $k' = k$ again and then for each k, obtaining

$$\boxed{\frac{d}{d\epsilon}\Big|_{\epsilon=0} a_k(1, D-\epsilon F) = a_{k-2}(F, D)}\ . \tag{146}$$

For the last one (58) we consider $D(\epsilon, \delta) = e^{-2\epsilon f}(D - \delta F)$. Next, using (56) with $n = k$ we have

$$\frac{d}{d\epsilon}\Big|_{\epsilon=0} a_n(1, D(\epsilon, \delta)) = 0\ , \tag{147}$$

so that introducing the variation over δ and using (57) in the second step

$$\boxed{0 = \frac{d}{d\epsilon}\Big|_{\epsilon=0} \frac{d}{d\delta}\Big|_{\delta=0} a_n(1, D(\epsilon, \delta)) = \frac{d}{d\epsilon}\Big|_{\epsilon=0} a_{k-2}(e^{-2\epsilon f} F, e^{-2\epsilon f} D)} \tag{148}$$

A.2 Local scale transformation for the heat kernel

In this appendix we want to focus on the relations obtained when permorming local scale transformations (77). First of all, in order to see how the required quantities transform, we recall the definiton of E and ω_δ

$$\begin{aligned}\omega_\delta &= \frac{1}{2}g_{\nu\delta}(a^\nu + g^{\mu\sigma}\Gamma^\nu_{\mu\sigma}\,\mathcal{I}_\mathcal{V})\ ,\\ E &= b - g^{\nu\mu}(\partial_\mu\omega_\nu + \omega_\nu\omega_\mu - \omega_\sigma\Gamma^\sigma_{\nu\mu})\ .\end{aligned} \tag{149}$$

Using the transformations of (65) we get

$$\begin{aligned}&\tilde{\Gamma}^m_{ij} = \Gamma^m_{ij} + \delta^m_i f_{;j}\epsilon + \delta^m_j f_{;i}\epsilon - f^{;m}g_{ij}\epsilon\ ,\\ &\tilde{\omega}_\delta = \omega_\delta + \frac{1}{2}(n-2)f_{;\delta}\epsilon\ ,\\ &\tilde{E} = e^{-2\epsilon f}E + e^{-2\epsilon f}g^{\mu\nu}\left(\frac{1}{2}(2-n)f_{;\nu\mu}\epsilon + \mathcal{O}(\epsilon^2)\right)\ ,\\ &\tilde{E}^2 = e^{-4\epsilon f}\left(E^2 + (n-2)\epsilon f^{;\mu}_{;\mu} + \mathcal{O}(\epsilon^2)\right)\ .\end{aligned} \tag{150}$$

The terms with ϵ^2 do not contribute when taking the derivative with respect to ϵ and evaluating it at $\epsilon = 0$, so that[7]

$$\begin{aligned}&\frac{d}{d\epsilon}\Big|_{\epsilon=0}\tilde{E} = -2fE + \frac{1}{2}(n-2)f^{;\mu}_{;\mu}\ ,\\ &\frac{d}{d\epsilon}\Big|_{\epsilon=0}\tilde{E}^2 = -4fE^2 + (n-2)f^{;\mu}_{;\mu}E\ .\end{aligned} \tag{151}$$

We can also compute the transformations for the Riemann and Ricci tensors and for the Ricci scalar

$$\begin{aligned}&\tilde{R}_{\mu\nu\rho\sigma} = e^{2\epsilon f}\left(g_{\mu\sigma}\epsilon f_{;\nu\rho} + g_{\nu\rho}\epsilon f_{;\mu\sigma} - g_{\nu\sigma}\epsilon f_{;\mu\rho} - g_{\mu\rho}\epsilon f_{;\nu\sigma}\right)\ ,\\ &\tilde{R}_{\nu\sigma} = \tilde{g}^{\mu\rho}\tilde{R}_{\mu\nu\rho\sigma} = R_{\nu\sigma} - (n-2)\epsilon f_{;\nu\sigma} - \epsilon g_{\nu\sigma}f^{;\mu}_{;\mu}\ ,\\ &\tilde{R} = \tilde{g}^{\nu\sigma}\tilde{R}_{\nu\sigma} = e^{-2\epsilon f}\left(R - 2(n-1)\epsilon f^{;\mu}_{;\mu}\right)\ ,\end{aligned} \tag{152}$$

so that the invariants formed with the contractions of two Ricci and Riemann tensors transform as

$$\begin{aligned}&\tilde{R}_{\mu\nu\rho\sigma}\tilde{R}^{\mu\nu\rho\sigma} = \tilde{R}_{\mu\nu\rho\sigma}\tilde{R}_{\alpha\beta\gamma\delta}\tilde{g}^{\mu\alpha}\tilde{g}^{\nu\beta}\tilde{g}^{\rho\gamma}\tilde{g}^{\sigma\delta} = e^{-4\epsilon f}\left(R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} - 8f_{;\mu\nu}R^{\mu\nu} + \mathcal{O}(\epsilon^2)\right)\ ,\\ &\tilde{R}_{\nu\sigma}\tilde{R}^{\nu\sigma} = \tilde{R_\nu\sigma}\tilde{R_\mu\rho}\tilde{g}^{\nu\mu}\tilde{g}^{\sigma\rho} = e^{-4\epsilon f}\left(R_{\nu\sigma}R^{\nu\sigma} - 2(n-2)\epsilon f_{;\nu\sigma}R^{\nu\sigma} - 2Rf^{;\nu}_{;\nu} + \mathcal{O}(\epsilon^2)\right)\ ,\\ &\tilde{R}^2 = e^{-4\epsilon f}\left(R^2 - 4(n-1)\epsilon f^{;\mu}_{;\mu}R + \mathcal{O}(\epsilon^2)\right)\ .\end{aligned} \tag{153}$$

[7]Note that we are using flat indices so that sumation is on repeated indices.

This implies,

$$\begin{aligned}
\frac{d}{d\epsilon}\Big|_{\epsilon=0}\tilde{R} &= -2fR - 2(n-1)f_{;\mu}^{;\mu} \ , \\
\frac{d}{d\epsilon}\Big|_{\epsilon=0}\tilde{R}^2 &= -4fR^2 - 4(n-1)f_{;\mu}^{;\mu}R \ , \\
\frac{d}{d\epsilon}\Big|_{\epsilon=0}\tilde{R}_{\mu\nu}\tilde{R}^{\mu\nu} &= -4fR_{\mu\nu}R^{\mu\nu} - 2f_{;\mu}^{;\mu}R - 2(n-2)f_{;\mu\nu}R^{\mu\nu} \ , \\
\frac{d}{d\epsilon}\Big|_{\epsilon=0}\tilde{R}_{\mu\nu\rho\sigma}\tilde{R}^{\mu\nu\rho\sigma} &= -4fR_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} - 8f_{;\mu\nu}R^{\mu\nu} \ .
\end{aligned} \tag{154}$$

The rest of transformations regarding the product of E and R and their derivatives are computed similarly. Finally, for the field strength tensor we have

$$\begin{aligned}
\tilde{\Omega}_{\mu\nu} = \Omega_{\mu\nu} \ \longrightarrow \ \tilde{\Omega}^2 &= \tilde{\Omega}_{\mu\nu}\tilde{\Omega}_{\rho\sigma}\tilde{g}^{\mu\rho}\tilde{g}^{\nu\sigma} = e^{-4\epsilon f}\Omega^2 \ , \\
\frac{d}{d\epsilon}\Big|_{\epsilon=0}\tilde{\Omega}^2 &= -4f\Omega^2 \ .
\end{aligned} \tag{155}$$

A.3 Computation of α_{10}

Here we compute α_{10} explicitly. To do that, we are using the heat kernel of $\mathcal{M} = \mathbb{R}^n$ with flat metric. Using the basis of the normalized plane waves we can write

$$\begin{aligned}
K(f;t) &= \mathrm{Tr}_{L^2}(f\exp(-tD)) = \int d^nx \int \frac{d^nk}{(2\pi)^n} e^{-ikx}\mathrm{Tr}_{\mathcal{V}}\left\{f(x)\exp(-tD)e^{ikx}\right\} = \\
&= \int d^nx \int \frac{d^nk}{(2\pi)^n}\mathrm{Tr}_{\mathcal{V}}\left\{f(x)\exp(t\left[(\nabla^\mu + ik^\mu)^2 + E\right])\right\} \ .
\end{aligned} \tag{156}$$

For the computation we need the following integrals

$$\begin{aligned}
\int \frac{d^nk}{(2\pi)^n}e^{-tk^2} &= \frac{1}{(4t\pi)^{n/2}} \ , \\
\int \frac{d^nk}{(2\pi)^n}e^{-tk^2}k^\mu k^\nu &= \frac{1}{(4t\pi)^{n/2}}\frac{1}{2t}g^{\mu\nu} \ , \\
\int \frac{d^nk}{(2\pi)^n}e^{-tk^2}k^\mu k^\nu k^\rho k^\sigma &= \frac{1}{(4t\pi)^{n/2}}\frac{1}{4t^2}\left(g^{\mu\nu}g^{\rho\sigma} + g^{\mu\rho}g^{\nu\sigma} + g^{\mu\sigma}g^{\nu\rho}\right) \ .
\end{aligned} \tag{157}$$

So what we do now is to isolate $\exp(-tk^2)$ in (156) and expand the rest of the exponential in a power series of t, giving

$$
\begin{aligned}
K(f;t) &= \int d^n x \int \frac{d^n k}{(2\pi)^n} e^{-tk^2} \mathrm{Tr}_{\mathcal{V}} \left\{ f(x) \left[1 + t(\nabla^2 + E) \right.\right. \\
&- \frac{t^2}{2} 4(k\nabla)^2 + \frac{t^2}{2} \left(\nabla^2\nabla^2 + \nabla^2 E + E\nabla^2 + E^2 \right) \\
&- \frac{4t^3}{6} \left((k\nabla)^2 E + E(k\nabla)^2 + (k\nabla)E(k\nabla) \right) \\
&- \frac{4t^3}{6} \left((k\nabla)^2 \nabla^2 + \nabla^2 (k\nabla)^2 + (k\nabla)\nabla^2 (k\nabla) \right) \\
&+ \left.\left. \frac{16t^4}{24} (k\nabla)^4 + \ldots \right] \right\} . \qquad (158)
\end{aligned}
$$

Using the integrals (157) we obtain

$$
\begin{aligned}
K(f;t) &= \frac{1}{(4\pi t)^{n/2}} \int d^n x \ \mathrm{Tr}_{\mathcal{V}} \left\{ f(x) \left[1 + tE + \frac{t^2}{2} \left(\nabla^2\nabla^2 + \nabla^2 E + E\nabla^2 + E^2 \right) \right.\right. \\
&- \frac{t^2}{3} \left(\nabla^2 E + E\nabla^2 + \nabla^\mu E \nabla_\mu \right) - \frac{t^2}{3} \left(2\nabla^2\nabla^2 + \nabla^\mu \nabla^2 \nabla_\mu \right) \\
&+ \left.\left. \frac{t^2}{6} \left(\nabla^\mu \nabla^\nu \nabla_\mu \nabla_\nu + \nabla^2\nabla^2 + \nabla^\mu \nabla^2 \nabla_\mu \right) + \mathcal{O}(t^3) \right] \right\} . \qquad (159)
\end{aligned}
$$

Ignoring the surface terms and combining the derivatives terms into a commutator which is by definition the field strength $\Omega_{\mu\nu}$ we get

$$
K(f;t) = \frac{1}{(4\pi t)^{n/2}} \int d^n x \ \mathrm{Tr}_{\mathcal{V}} \left\{ f(x) \left[1 + tE + t^2 \left(\frac{1}{2} E^2 + \frac{1}{6} E_{;\mu}^{;\mu} + \frac{1}{12} \Omega_{\mu\nu} \Omega^{\mu\nu} \right) + \mathcal{O}(t^3) \right] \right\} . \qquad (160)
$$

From this equation we obtain that $\boxed{\alpha_{10} = 30}$ and also a crosscheck for other coefficients such as α_0, α_1, α_3 and α_5.

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