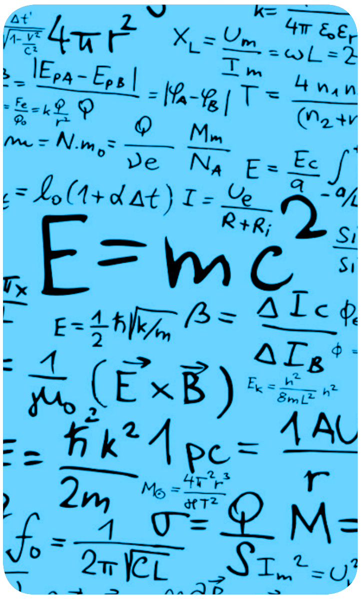

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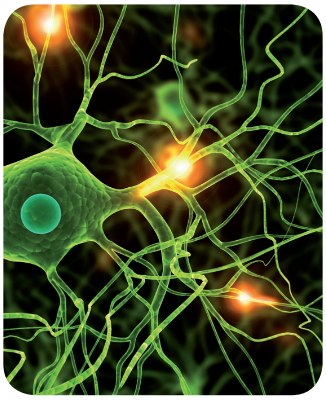
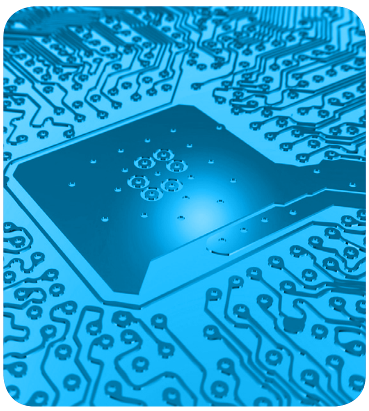
Local and Global Analysis of a Non-linear Schrödinger Equation on R³

Víctor Arnaiz Solórzano





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*El nuestro es un oficio
de galeotes, no de diletantes.*

Gabriel García Márquez

Contents

§1. Introduction

¿De do viene una cosa, que, si fuera
menos veces de mí probada y vista,
según parece que a razón resista,
a mi sentido mismo no creyera?

Garcilaso de la Vega

When Garcilaso was writing these lines, he was for sure not thinking about the theme of this work, but the reader will maybe recognize the parallelism: how difficult it is to understand the deepest human passions, no matter the times we come across them; and how far seem modern physical theories to be away from intuition, even for the ones familiar with their mathematical formulations. This is well illustrated by the paradigmatic example of quantum mechanics, and more specifically by its master equation, Schrödinger's equation.

In this survey we study the nonlinear Schrödinger equation (NLS). We study the subject from a purely mathematical point of view. Although we take the chance to expose several examples in physics where the NLS appears (see [18]), such as laser beam propagation in a medium which index of refraction is sensitive to the wave amplitude, water waves at the free surface of an ideal fluid, and plasma waves. To be precise, the considered equation is the elliptic NLS, noting that the differential operator involved is the laplacian. The elliptic NLS arises also in other contexts. In quantum mechanics, it is obtained in localizing the potential of the Hartree equation. In chemistry, it appears as a continuous-limit model for mesoscopic molecular structures.

Our first aim is to understand local-existence theory and long time behavior for the solutions of the NLS. This theory has been extensively developed by Cazenave [3], Ginibre [10] and Strauss [16]. In this first part our principal reference has been the book of Linares and Ponce [13], where they introduce basic results of local-existence, conservation laws and long-time behavior for the solutions of the initial value problem (IVP)

$$(1.1)\qquad \begin{cases} i\partial_t u(x,t) + \Delta u(x,t) = -\lambda |u(x,t)|^{\alpha-1} u, & x \in \mathbb{R}^n, t \in \mathbb{R} \\ u(x,0) = u_0(x), \end{cases}$$

for admissible values of n, α, where λ takes the values $+1$, focusing case, or -1, defocusing case; and where u_0 belongs to different functional spaces as $L^2(\mathbb{R}^n)$ and Sobolev spaces $H^k(\mathbb{R}^n)$ for some values of k. We also had to study basic properties of the solutions of the linear Schrödinger equation, as well as estimates for the solution operator, in particular *Strichartz type estimates.*

The second and most important part of the work is about the article writen by Colliander, Keel, Staffilani, Takaoka and Tao [7]. In this article it is shown global-existence and scattering for rough solutions of this particular case of the equation (1.1),

$$i\partial_t\phi(x,t) + \Delta\phi(x,t) = |\phi(x,t)|^2\phi, \quad x \in \mathbb{R}^3,\, t \geq 0. \tag{1.2}$$

In this context, rough means that we are considering solutions of (1.2) with initial data $\phi(x,0) = \phi_0(x)$ very irregular; more precisely, ϕ_0 is in the Sobolev space $H^s(\mathbb{R}^n)$ for $s < 1$. Scattering means that the solutions of the NLS, under certain conditions, behave as solutions of the linear Schrödinger equation for long-time evolution. This type of solutions are so called dispersive solutions (see [18]). In [4], Cazenave proves local-existence for (1.2) in $H^s(\mathbb{R}^3)$ for $s > 1/2$. In [7] they prove global-existence and scattering for solutions with initial data in $H^s(\mathbb{R}^n)$ for $s > 4/5$. This is the best result known, but it is not expected to be optimal. This result may possibly be improved up to $s = 1/2$, which is the critical exponent where the H^s-norm of the solutions is invariant by rescaling. Some previous results are those of Ginibre and Velo [11], where they show scattering in $H^1(\mathbb{R}^n)$; some works of Bourgain [1], [2], where he proves global existence for $s > 11/13$ in the general case, and for $s > 5/7$ in the case of radially-symmetric solutions; and a previous article of the authors (of [7]) [8], where they prove global existence for $s > 5/6$ without scattering.

Now we introduce the main tools we have used. First of all, the Strichartz estimates have been needed in local-existence proofs. This type of estimates appears for the first time in works of Strichartz [17] which were motivated by Stein. Strichartz proves inequalities of the form

$$\| \int e^{i(x\xi + t\phi(\xi))} g(\xi) d\xi \|_{L^q(\mathbb{R}^{n+1})} \leq C \|g\|_{L^2(\mathbb{R}^n)},$$

for some particular ϕ ($\phi = |\cdot|^2$, Schrödinger equation; $\phi = |\cdot|$, wave equation). These inequalities are related to restriction problems, in which attention is paid to the boundedness of operators of the form

$$\begin{array}{rccc} \mathcal{R}: & L^a(\mathbb{R}^{n+1}) & \to & L^b(S) \\ & f & \to & \widehat{f}|_S, \end{array}$$

where S is a surface embedded in $\mathbb{R}^{n+1}$. The adjoint $\mathcal{R}^*$ is the solution operator (transformed by Fourier) of a certain PDE determined by the surface S considered. In the case of Schrödinger's equation, S is the paraboloid $(\xi, |\xi|^2)$, while for the wave equation, the surface is the cone $(\xi, |\xi|)$.

Another main tool used in [7] (preceded by some ideas of Bourgain [1], [2]) and successfully used in (2000-2005) for its authors, is to introduce an operator of the form $\widehat{I\phi} = m\widehat{\phi}$ which acts on the solution ϕ of the problem considered, acting as the identity on the low frequencies and adding an extra decay on the high frequencies (increasing the regularity of ϕ). This technique allows obtaining almost conservation laws, that is, the laws that control the energy variation of the quasi-solutions as time increases, and obtaining hidden information about the original solutions, since these solutions have infinite energy.

The main new contribution in [7] is an improvement of the so called *Morawetz estimates*, which allows them to estimate the norm $L^4_{t,x}(\mathbb{R}^3, \mathbb{R})$ of the solutions, a key ingredient in the proof of the global-existence result.

Morawetz estimates are based on the intrinsic geometry of the equation. It is analyzed the L^2 mass current which scatters radially with respect to every point of the space. The average of the radial component of the mass current is named *Morawetz action*, and it is expected to increase if the wave scatters since such behavior involves a broadening redistribution of the L^2 mass. For the equation (1.2), Lin and Strauss proved that the repulsion condition $\lambda = -1$ and the form of the nonlinear potential are sufficient to obtain that the derivative of the Morawetz action is positive. The improvement introduced in [7] consist on average the Morawetz action with respect to the L^2 mass. This new object is called Morawetz Interaction-Potential, and this allows us to estimate the norm mentioned above.

At the end of the work we include an appendix with some topics about harmonic analysis which have been used systematically. These topics are: the Fourier transform, the Calderon Zygmund theorem, the Hardy-Littlewood-Soboled theorem, the Littlewood-Paley theory, Sobolev spaces and fractional Leibniz rules.

Remark of notation used.

Given $A, B \geq 0$, we'll write $A \lesssim B$ meaning that for some universal constant $K > 2$, $A \leq K \cdot B$. We'll write $A \sim B$ when both $A \lesssim B$ and $B \lesssim A$.

§2. The Linear Schrödinger Equation

We start by studing the Initial Value Problem (IVP)

$$\left\{\begin{array}{l} \partial_t u = i\Delta u + F(x,t), \quad x\in\mathbb{R}^n, \;\; t\in\mathbb{R}, \\ u(x,0)=u_0(x). \end{array}\right. \tag{2.1}$$

We are interested in solving (2.1) with initial data $u_0 \in L^2(\mathbb{R}^n)$ and in the Sobolev spaces $H^s(\mathbb{R}^n)$, with $s > 0$. Moreover, we analyze some properties of the solutions of (2.1) and obtain Strichartz type estimates for the solutions.

2.1. Basic Results

We consider the (IVP) for the Free Schrödinger Equation

$$\left\{\begin{array}{l} \partial_t u = i\Delta u, \quad x\in\mathbb{R}^n, \; t\in\mathbb{R}, \\ u(x,0)=u_0(x), \end{array}\right. \tag{2.2}$$

To find the solution, we need the following proposition.

Proposition 2.1. *Let $a \neq 0$ a complex number with $\mathscr{R}ea \geq 0$, and consider the distribution $e^{-a|\cdot|^2} \in \mathscr{S}'(\mathbb{R}^n)$. Then, the Fourier transform of $e^{-a|\cdot|^2}$ is given by*

$$(\widehat{e^{-a|\cdot|^2}})(\xi) = \left(\frac{\pi}{a}\right)^{n/2} e^{-\pi^2|\xi|^2/a},$$

with $\sqrt{z}$ definided in $\mathbb{C}\setminus\{\mathscr{R}ez<0\}$.

Proof. We assume first that $\mathscr{R}ea > 0$, then $e^{-a|\cdot|^2} \in L^1(\mathbb{R}^n)$. By the definition of the Fourier transform, we obtain

$$\begin{aligned} \int_{\mathbb{R}^n} e^{-a|x|^2} e^{-2\pi i(x\cdot\xi)} dx &= \prod_{j=1}^n \int_{-\infty}^{\infty} e^{(-ax_j^2-2\pi i\xi_j x_j)} dx_j \\ &= \prod_{j=1}^n \int_{-\infty}^{\infty} e^{(-ax_j^2-2\pi i\xi_j x_j+\pi^2\xi_j^2/a)} dx_j e^{-\pi^2\xi_j^2/a} \\ &= \prod_{j=1}^n e^{-\pi^2\xi_j^2/a} \int_{-\infty}^{\infty} e^{-(\sqrt{a}x_j+i\pi\xi_j/\sqrt{a})^2} dx_j \\ &= \left(\frac{\pi}{a}\right)^{n/2} e^{-\pi^2|\xi|^2/a}, \end{aligned}$$

where the last equality is given by the following complex calculus, which is justified by the Cauchy theorem and an argument based on taking the limit on the path of integration.

$$\int_{-\infty}^{\infty} e^{-(\sqrt{a}x+i\pi\xi/\sqrt{a})^2}dx = \frac{1}{\sqrt{a}}\int_{-\infty}^{\infty} e^{-y^2}dy = \sqrt{\frac{\pi}{a}}.$$

If $\mathscr{R}ea = 0$, then $e^{-a|\cdot|^2} \notin L^1(\mathbb{R}^n)$ and we have to prove the result in the sense of the temperated distributions, that is to say, given $\varphi \in \mathscr{S}(\mathbb{R}^n)$, it must satisfy

$$\int_{\mathbb{R}^n} e^{-a|x|^2}\widehat{\varphi}(x)dx = \left(\frac{\pi}{a}\right)^{n/2}\int_{\mathbb{R}^n} e^{-\pi^2|x|^2/a}\varphi(x)dx. \tag{2.3}$$

Let $0 < \varepsilon \to 0$, then, by the dominated convergence theorem, the left hand side of (2.3) is equal to

$$\begin{aligned}\lim_{\varepsilon\to 0}\int_{\mathbb{R}^n} e^{-(a+\varepsilon)|x|^2}\widehat{\varphi}(x)dx &= \lim_{\varepsilon\to 0}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} e^{-(a+\varepsilon)|x|^2}e^{-2\pi i(x\cdot y)}dx\,\varphi(y)dy\\ &= \lim_{\varepsilon\to 0}\int_{\mathbb{R}^n}\left(\frac{\pi}{a+\varepsilon}\right)^{n/2} e^{\frac{-\pi^2|y|^2}{a+\varepsilon}}\varphi(y)dy\end{aligned}$$

and, again by dominated convergence, it follows (2.3). □

To solve (2.2), taking the Fourier transform in the spatial variable x, we can write

$$\begin{cases}\widehat{\partial_t u}(\xi,t) = \partial_t\widehat{u}(\xi,t) = \widehat{i\Delta u}(\xi,t) = -4\pi^2 i|\xi|^2\widehat{u}(\xi,t)\\ \widehat{u}(\xi,0) = \widehat{u}_0(\xi).\end{cases}$$

The solution of this ordinary differential equation is

$$\widehat{u}(\xi,t) = e^{-4\pi^2 it|\xi|^2}\widehat{u}_0(\xi).$$

By the proposition 2.1, it follows that

$$\begin{aligned}u(x,t) &= (e^{-4\pi^2 it|\xi|^2}\widehat{u}_0(\xi))^\vee = (e^{-4\pi^2 it|\xi|^2})^\vee * u_0(x)\\ &= \frac{e^{i|\cdot|^2/4t}}{(4\pi it)^{n/2}} * u_0(x) = e^{it\Delta}u_0(x),\end{aligned}$$

where we have introduced the notation $e^{it\Delta}$, which will be justified soon.

Now we include a list of some solutions of (2.2) obtained by the invariance properties of the equation.

Proposition 2.2. *If $u = u(x,t)$ is a solution of* (2.2)*, then*

$u_1(x,t) = e^{i\theta}u(x,t),\quad \theta\in\mathbb{R}$ *fixed,*

$u_2(x,t) = u(x-x_0,t-t_0),$ *with* $x_0\in\mathbb{R}^n,\ t_0\in\mathbb{R}$ *fixed,*

$u_3(x,t) = u(Ax,t),$ *with A any orthogonal matrix $n\times n$,*

$u_4(x,t) = u(x-2x_0t,t)e^{i(x\cdot x_0-|x_0|^2t)},$ *with* $x_0\in\mathbb{R}^n$ *fixed,*

$u_5(x,t) = \lambda^{n/2}u(\lambda x,\lambda^2 t),\ \lambda\in\mathbb{R}$ *fixed,*

$u_6(x,t) = \dfrac{1}{(\alpha+\omega t)^{n/2}}\exp\left[\dfrac{i\omega|x|^2}{4(\alpha+\omega t)}\right]u\left(\dfrac{x}{\alpha+\omega t},\dfrac{\gamma+\theta t}{\alpha+\omega t}\right),\quad \alpha\theta-\omega\gamma = 1,$

also satisfy the equation (2.2)*.*

The next properties of the family of operators $\{e^{it\Delta}\}_{t=-\infty}^{\infty}$ justify this notation.

Proposition 2.3.

1. *For all* $t \in \mathbb{R}$, $e^{it\Delta} : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$ *is an isometry; which implies*
$$\|e^{it\Delta} f\|_2 = \|f\|_2.$$
2. $e^{it\Delta}e^{it'\Delta} = e^{i(t+t')\Delta}$ *with* $(e^{it\Delta})^{-1} = e^{-it\Delta} = (e^{it\Delta})^*$.
3. $e^{i0\Delta} = 1$.
4. *Fixing* $f \in L^2(\mathbb{R}^n)$, *the function* $\Phi_f : \mathbb{R} \to L^2(\mathbb{R}^n)$ *defined by* $\Phi_f(t) = e^{it\Delta} f$ *is a continuous function; i.e., it describes a curve in* $L^2(\mathbb{R}^n)$.

These four properties allow us to name the family $\{e^{it\Delta}\}_{t=-\infty}^{\infty}$ a *unitary group of operators* in the Hilbert space $L^2(\mathbb{R}^n)$. Moreover, the family $\{e^{it\Delta}\}_{t=-\infty}^{\infty}$ is also a unitary group in $H^s(\mathbb{R}^n)$ for all $s \in \mathbb{R}$, since
$$\|e^{it\Delta} f\|_{H^s(\mathbb{R}^n)} = \|\langle\nabla\rangle^s(e^{it\Delta} f)\|_2 = \|e^{it\Delta}(\langle\nabla\rangle^s f)\|_2 = \|\langle\nabla\rangle^s f\|_2 = \|f\|_{H^s(\mathbb{R}^n)}.$$
The other properties follows quickly from this.

Now we establish some properties of the group $\{e^{it\Delta}\}_{t=-\infty}^{\infty}$ in the $L^p(\mathbb{R}^n)$-spaces.

Lemma 2.1. *If* $t \neq 0$, $\frac{1}{p} + \frac{1}{p'} = 1$ *and* $p' \in [1, 2]$, *then we have* $e^{it\Delta} : L^{p'}(\mathbb{R}^n) \to L^p(\mathbb{R}^n)$ *is continuous and*
$$\|e^{it\Delta} f\|_p \leq c|t|^{-n/2(1/p'-1/p)}\|f\|_{p'}.$$

Proof. We know, from Proposition 2.3, that $e^{it\Delta}$ is an isometry in $L^2(\mathbb{R}^n)$; that is
$$\|e^{it\Delta} f\|_2 = \|f\|_2.$$
By using Young's inequality, we also have
$$\begin{aligned}\|e^{it\Delta} f\|_\infty &= \|\frac{e^{i|\cdot|^2/4t}}{(4\pi it)^{n/2}} * f\|_\infty \\ &\leq \|\frac{e^{i|\cdot|^2/4t}}{(4\pi it)^{n/2}}\|_\infty \|f\|_1 \leq c|t|^{-n/2}\|f\|_1.\end{aligned}$$
Applying Riesz-Thorin interpolation theorem, we obtain
$$e^{it\Delta} : L^{p'}(\mathbb{R}^n) \to L^p(\mathbb{R}^n), \quad \text{with } \frac{1}{p} + \frac{1}{p'} = 1,$$
and
$$\|e^{it\Delta} f\|_p \leq (c|t|^{-n/2})^{1-\theta}\|f\|_{p'}$$
where
$$\frac{1}{p} = \frac{\theta}{2} \text{ and } 1 - \theta = 1 - \frac{2}{p} = \frac{1}{p'} - \frac{1}{p}.$$
This completes the proof of the lemma. $\square$

Finally, we prove the Duhamel's principle, providing the solution of the IVP (2.1), giving by the formula

$$u(x,t) = e^{it\Delta}u_0 + \int_0^t e^{i(t-t')\Delta}F(\cdot,t')dt', \tag{2.4}$$

for $f \in \mathcal{C}(\mathbb{R} : \mathscr{S}(\mathbb{R}^n))$. In fact, the two summands of the formula above correspond to the homogenous solution of (2.2) and the non-homogenuous solution with zero data respectively. So we only have to show that

$$\begin{aligned}\partial_t\left(\int_0^t e^{i(t-t')\Delta}F(\cdot,t')dt'\right) &= \int_0^t \partial_{t'}(e^{i(t-t')\Delta}F(\cdot,t'))dt' + \int_0^t \partial_t(e^{i(t-t')\Delta}F(\cdot,t'))dt' \\ &= F(x,t) + i\Delta\left(\int_0^t e^{i(t-t')\Delta}F(\cdot,t')dt'\right),\end{aligned}$$

and we are done.

2.2. Strichartz-type Estimates

The result proved in this section describes the *global smoothing* properties of the group $\{e^{it\Delta}\}_{t=-\infty}^{\infty}$.

Theorem 2.1 ([13], page 64). *The group $\{e^{it\Delta}\}_{t=-\infty}^{\infty}$ satisfies:*

$$\left(\int_{-\infty}^{\infty} \|e^{it\Delta}f\|_p^q dt\right)^{1/q} \leq c\|f\|_2, \tag{2.5}$$

$$\left(\int_{-\infty}^{\infty} \|\int_{-\infty}^{\infty} e^{i(t-t')\Delta}g(\cdot,t')dt'\|_p^q dt\right)^{1/q} \leq c\left(\int_{-\infty}^{\infty} \|g(\cdot,t)\|_{p'}^{q'}dt\right)^{1/q'}, \tag{2.6}$$

and

$$\|\int_{-\infty}^{\infty} e^{it\Delta}g(\cdot,t)dt\|_2 \leq c\left(\int_{-\infty}^{\infty} \|g(\cdot,t)\|_{p'}^{q'}dt\right)^{1/q'}, \tag{2.7}$$

with

$$\left.\begin{array}{ll} 2 \leq p < \dfrac{2n}{n-2} & \textit{if } n \geq 3 \\ 2 \leq p < \infty & \textit{if } n = 2 \\ 2 \leq p \leq \infty & \textit{if } n = 1 \end{array}\right\} \quad \textit{and} \quad \frac{2}{q} = \frac{n}{2} - \frac{n}{p}, \tag{2.8}$$

where $c = c(p,n)$ is a constant that depends only on p and n, and $\frac{1}{p} + \frac{1}{p'} = \frac{1}{q} + \frac{1}{q'} = 1$.

Proof. First we show that the three inequalities (2.5), (2.6) and (2.7) are equivalent. By duality, Fubini's Theorem and Hölder's inequality, we have

$$\begin{aligned}\|\int_{-\infty}^{\infty} e^{it\Delta}g(\cdot,t)dt\|_2 &= \sup\left\{\left|\int_{\mathbb{R}^n} f(x)\left(\int_{-\infty}^{\infty} e^{it\Delta}g(x,t)dt\right)dx\right| : \|f\|_2 = 1\right\} \\ &= \sup\left\{\left|\int_{-\infty}^{\infty}\int_{\mathbb{R}^n} (e^{it\Delta}f)(x)g(x,t)\,dx\,dt\right| : \|f\|_2 = 1\right\} \\ &\leq \sup\left\{\left(\int_{-\infty}^{\infty} \|e^{it\Delta}f\|_p^q dt\right)^{1/q}\left(\int_{-\infty}^{\infty} \|g(\cdot,t)\|_{p'}^{q'}dt\right)^{1/q'} : \|f\|_2 = 1\right\},\end{aligned}$$

then (2.5) implies (2.7). Analogously, we obtain that (2.7) implies (2.5). To show that (2.6) and (2.7) are equivalent, we use that

$$\begin{aligned}\|\int_{-\infty}^{\infty} e^{it\Delta}g(\cdot,t)dt\|_2^2 &= \int_{\mathbb{R}^n}\left(\int_{-\infty}^{\infty} e^{it\Delta}g(\cdot,t)dt\right)\overline{\left(\int_{-\infty}^{\infty} e^{it'\Delta}g(\cdot,t')dt'\right)}dx\\ &= \int_{\mathbb{R}^n}\int_{-\infty}^{\infty} g(x,t)\left(\int_{-\infty}^{\infty} e^{i(t-t')\Delta}\overline{g(\cdot,t')}dt'\right)dt\,dx,\end{aligned}$$

and again duality and Hölder's inequality.

It remains to prove (2.6). By Minkowski's inequality and Lemma 2.1,

$$\begin{aligned}\|\int_{-\infty}^{\infty} e^{i(t-t')\Delta}g(\cdot,t')dt'\|_p &\le \int_{-\infty}^{\infty}\|e^{i(t-t')\Delta}g(\cdot,t')\|_p dt'\\ &\le c\int_{-\infty}^{\infty}\frac{1}{|t-t'|^{\alpha}}\|g(\cdot,t')\|_{p'}dt'\end{aligned}$$

with $\alpha = (n/2)(1/p' - 1/p)$. Now, using this inequality and Theorem A.7 (Hardy-Littlewood-Sobolev) we conclude that

$$\begin{aligned}&\left(\int_{-\infty}^{\infty}\|\int_{-\infty}^{\infty} e^{i(t-t')\Delta}g(\cdot,t')dt'\|_p^q dt\right)^{1/q}\\ &\qquad\le c\|\int_{-\infty}^{\infty}\frac{1}{|t-t'|^{\alpha}}\|g(\cdot,t')\|_{p'}dt'\|_q \le c\left(\int_{-\infty}^{\infty}\|g(\cdot,t)\|_{p'}^{q'}dt\right)^{1/q'}\end{aligned}$$

with $1/q' = 1/q + (1-\alpha)$ and $0 < 1-\alpha < 1$, that is, $n/2 = 2/q + n/p$, where

$$\left.\begin{array}{ll} 2 \le p < \dfrac{2n}{n-2} & \text{if } n \ge 3\\ 2 \le p < \infty & \text{if } n = 2\\ 2 \le p \le \infty & \text{if } n = 1\end{array}\right\}.$$

This concludes the proof.

□

A more general statement may be established. The following corollary will be used in the proof of the local-existence result in the next section.

Corollary 2.1 ([13], page 66). *Let* (p_0, q_0), $(p_1, q_1) \in \mathbb{R}^2$ *satisfying the condition* (2.8). *Then for all* $T > 0$ *we have*

$$\left(\int_0^T \|\int_0^t e^{i(t-t')\Delta}g(\cdot,t')dt'\|_{p_1}^{q_1}dt\right)^{1/q_1} dt \le c\left(\int_0^T \|g(\cdot,t)\|_{p_0'}^{q_0'}dt\right)^{1/q_0'}, \tag{2.9}$$

with $c = c(n, p_0, p_1)$.

Proof. Assume, without loss of generality, that $p_0 \in [2, p_1)$. Using inequality (2.6) with integration indices $0, T$ and $0, t$ instead of $-\infty, \infty$ (the proof is completely analogous) we have

$$\left(\int_0^T \|\int_0^t e^{i(t-t')\Delta}g(\cdot,t')dt'\|_{p_1}^{q_1}dt\right)^{1/q_1} \le c\left(\int_0^T \|g(\cdot,t')\|_{p_1'}^{q_1'}dt\right)^{1/q_1'}.$$

Similarly, by using inequality (2.7),

$$\sup_{[0,T]} \| \int_0^t e^{i(t-t')\Delta} g(\cdot,t')dt' \|_2 = \sup_{[0,T]} \| e^{it\Delta} \int_0^t e^{-it'\Delta} g(\cdot,t')dt' \|_2$$

$$= \sup_{[0,T]} \| \int_0^t e^{-it'\Delta} g(\cdot,t')dt' \|_2 \le c \left(\int_0^T \|g(\cdot,t)\|_{p_1'}^{q_1'} dt \right)^{1/q_1'} .$$

Now, note that if $\frac{1}{p_0} = \frac{\theta}{2} + \frac{1-\theta}{p_1}$ then $\frac{1}{q_0} = \frac{\theta}{\infty} + \frac{1-\theta}{q_1}$, since the pairs (p_0, q_0) and (p_1, q_1) satisfy the condition (2.8). Then, aplying Hölder's inequality and the previous inequalities we obtain

$$\begin{aligned}
&\left(\int_0^T \| \int_0^t e^{i(t-t')\Delta} g(\cdot,t')dt' \|_{p_0}^{q_0} dt \right)^{1/q_0} \\
&\quad \le \left(\int_0^T \| \int_0^t e^{i(t-t')\Delta} g(\cdot,t')dt' \|_2^{\theta q_0} \| \| \int_0^t e^{i(t-t')\Delta} g(\cdot,t')dt' \|_{p_1}^{(1-\theta)q_0} dt \right)^{1/q_0} \\
&\quad \le c \left(\int_0^T \|g(\cdot,t)\|_{p_1'}^{q_1'} dt \right)^{\theta/q_1'} \left(\int_0^T \| \int_0^t e^{i(t-t')\Delta} g(\cdot,t')dt' \|_{p_1}^{q_1} dt \right)^{(1-\theta)/q_1} \\
&\quad \le c \left(\int_0^T \|g(\cdot,t)\|_{p_1'}^{q_1'} dt \right)^{1/q_1'} .
\end{aligned}$$

To finish the proof, an argument of duality allows us to write the inequality

$$\left(\int_0^T \| \int_0^t e^{i(t-t')\Delta} g(\cdot,t')dt' \|_{p_1}^{q_1} dt \right)^{1/q_1} \le c \left(\int_0^T \|g(\cdot,t)\|_{p_0'}^{q_0'} dt \right)^{1/q_0'} .$$

This yields the result. □

§3. The Nonlinear Schrödinger Equation

In the first part of this section we shall estudy local well-posedness of the nonlinear IVP,

$$
\begin{cases} i\partial_t u = -\Delta u - \lambda |u|^{\alpha-1}u, \\ u(x,0) = u_0(x), \end{cases} \tag{3.1}
$$

$t \in \mathbb{R}$, $x \in \mathbb{R}^n$, where λ and α are real constants with $\alpha > 1$. In particular, we'll give a detailed proof of the local-existence result for the cubic NLS equation on $\mathbb{R}^3$ that we'll need in the next section.

We'll also show classical conservation laws for nonlinear Schrödinger equations. Specifically, we'll prove L^2-norm conservation and energy conservation for solutions in H^1.

In the last part of the section, we'll show some examples of global existence and others of blow-up that occurs in different cases of nonlinear Schrödinger equations.

3.1. Local Theory

We consider the integral equation (see (2.4))

$$
u(t) = e^{it\Delta}u_0 + i\lambda \int_0^t e^{i(t-t')\Delta}(|u|^{\alpha-1}u)(t')\, dt'. \tag{3.2}
$$

The difference between this equation and the one in (3.1) is that this one does not require any differentiability on the solution. It is easy to see that if u is a solution of the differential equation (3.1) then it is also a solution of (3.2).

Definition 3.1. *We will say that the integral equation* (3.2) *is locally well-posed in a function space X, if for every $u_0 \in X$ there exists a time $T > 0$ and a unique solution $u \in \mathcal{C}([0,T); X) \cap \cdots$ of* (3.2) *for $(x,t) \in \mathbb{R}^n \times [0,T)$. Moreover, the map $u_0 \mapsto u(\cdot,t)$ locally defined from X to $\mathcal{C}([0,T); X)$ is continuous.*

We shall distinguish between the subcritical case, when $T = T(\|u_0\|_X) > 0$, and the critical case, when $T = T(u_0) > 0$.

Now we enunciate the subcritical existence theorems in the spaces H^1 and H^2 and how the regularity of the initial data is preserved for the solution. The proofs of these results are based on the contraction mapping principle. The general theory is described in [13].

Theorem 3.1 ([13], page 99). *Let $u_0 \in H^1(\mathbb{R}^n)$ and α satisfying*

$$\begin{cases} 1 < \alpha < \dfrac{n+2}{n-2}, & \text{if } n > 2, \\ 1 < \alpha < \infty, & \text{if } n = 1, 2. \end{cases} \tag{3.3}$$

Then there exist $T = T(\|u_0\|_{H^1(\mathbb{R}^n)}, n, \lambda, \alpha) > 0$ and a unique solution u of the integral equation (3.2) *in the interval* $[0, T)$ *with*

$$u \in \mathcal{C}([0,T) : H^1(\mathbb{R}^n)) \cap L^q([0,T) : W^{1,p}(\mathbb{R}^n)),$$

for all pairs (p, q) defined by condition (2.8)*.*

Moreover, for all $T' < T$ there exists a neighborhood W of u_0 in $H^1(\mathbb{R}^n)$ such that the function

$$\begin{array}{rcl} \mathbb{F} : W & \longrightarrow & \mathcal{C}([0,T'] : H^1(\mathbb{R}^n)) \cap L^q([0,T'] : W^{1,p}(\mathbb{R}^n)) \\ \widetilde{u}_0 & \longmapsto & \widetilde{u}(t) \end{array}$$

is Lipschitz.

Theorem 3.2 ([13], page 103). *Let $u_0 \in H^2(\mathbb{R}^n)$ and assume that the nonlinearity α satisfies*

$$\begin{cases} 2 \le \alpha < \dfrac{n}{n-4}, & \text{if } n \ge 5, \\ 2 \le \alpha < \infty, & \text{if } n \le 4. \end{cases} \tag{3.4}$$

Then there exist $T = T(\|u_0\|_{H^2(\mathbb{R}^n)}, n, \lambda, \alpha) > 0$ and a unique solution u of the integral equation (3.2) *in the interval of time* $[0, T)$ *with*

$$u \in \mathcal{C}([0,T) : H^2(\mathbb{R}^n)) \cap L^q([0,T) : W^{2,p}(\mathbb{R}^n)),$$

for all pairs (p, q) defined by condition (2.8)*.*

Moreover, for all $T' < T$ there exists a neighborhood W of u_0 in $H^2(\mathbb{R}^n)$ such that the function

$$\begin{array}{rcl} \mathbb{F} : W & \longrightarrow & \mathcal{C}([0,T'] : H^1(\mathbb{R}^n)) \cap L^q([0,T'] : W^{2,p}(\mathbb{R}^n)) \\ \widetilde{u}_0 & \longmapsto & \widetilde{u}(t) \end{array}$$

is Lipschitz.

As a consequence of the Theorem 3.2 we obtain the following relation between the differential equation (3.1) and the integral equation (3.2).

Corollary 3.2 ([13], page 103). *If u is the solution of equation* (3.2) *obtained in Theorem* 3.2*, then for all pair (p, q) which verifies condition* (2.8) *we have*

$$\partial_t u \in L^q([0,T) : L^p(\mathbb{R}^n)).$$

Moreover, u is the unique solution of the differential equation (3.1) in the time interval $[0, T)$*.*

These results show us how the regularity of the initial data is preserved with time. Now we prove a particular result, for the cubic defocusing NLS on $\mathbb{R}^3$, of the more general local-existence theorem proven by Cazenave and Weissler in [4]. This result will be important in the next section in order to prove global well-posedness.

Theorem 3.3. *Consider the IVP*

$$i\partial_t\phi(x,t)+\Delta\phi(x,t)=|\phi(x,t)|^2\phi(x,t),\quad x\in\mathbb{R}^3,\ t\geq 0, \tag{3.5}$$

$$\phi(x,0)=\phi_0(x)\in H^s(\mathbb{R}^3). \tag{3.6}$$

Then for all $s>\frac{1}{2}$ there exist $T=T(\|\phi_0\|_{H^s(\mathbb{R}^n)})>0$ and a unique solution ϕ of the integral equation (3.2) in the interval of time $[0,T)$ with

$$\phi\in\mathcal{C}([0,T);H^s(\mathbb{R}^n))\cap L^q([0,T):W^{s,p}(\mathbb{R}^n)),$$

for all pairs (p,q) defined by condition (2.8), that is rewriten in this case as

$$\frac{1}{q}+\frac{3}{2p}=\frac{3}{4},\quad \textit{with } 2\leq p<6. \tag{3.7}$$

Moreover, for all $T'<T$ there exists a neighborhood W of ϕ_0 in $H^s(\mathbb{R}^n)$ such that the function

$$\begin{array}{rcl}\mathbb{F}:W & \longrightarrow & \mathcal{C}([0,T']:H^s(\mathbb{R}^n))\cap L^q([0,T']:W^{s,p}(\mathbb{R}^n))\\ \widetilde{\phi}_0 & \longmapsto & \widetilde{\phi}(t)\end{array}$$

is Lipschitz.

Proof. In this case, the integral equation (3.2) takes the form

$$\phi(x,t)=e^{it\Delta}\phi_0-i\int_0^t e^{i(t-t')\Delta}(|\phi|^2\phi)(t')dt'. \tag{3.8}$$

By the Strichartz inequality (2.5) we have

$$\|\langle\nabla\rangle^s e^{it\Delta}\phi_0\|_{L_t^qL_x^p([0,T)\times\mathbb{R}^3)}\lesssim\|\phi_0\|_{H^s(\mathbb{R}^3)}$$

for all pair (p,q) satisfying condition (3.7), and by the properties of the group $\{e^{it\Delta}\}_{t=-\infty}^{\infty}$ we have

$$\sup_{[0,T)}\|e^{it\Delta}\phi_0\|_{H^s(\mathbb{R}^3)}=\|\phi_0\|_{H^s(\mathbb{R}^3)}.$$

Choose the admissible pairs $(4,\frac{8}{3})$ and $(\frac{12}{5},8)$. Now define

$$K_T=\|e^{it\Delta}\phi_0\|_{L_t^{8/3}W_x^{s,4}([0,T)\times\mathbb{R}^3)}+\|e^{it\Delta}\phi_0\|_{L_t^8W_x^{s,12/5}([0,T)\times\mathbb{R}^3)}.$$

Note that $K_T\to 0$, as $T\to 0$. Define also

$$X=\{u\in L_t^{8/3}W_x^{s,4}\cap L_t^8W_x^{s,12/5}([0,T)\times\mathbb{R}^3):\|u\|_{L_t^{8/3}W_x^{s,4}}+\|u\|_{L_t^8W_x^{s,12/5}}\leq 2K_T\}.$$

Denote

$$\Phi_u(t)=-i\int_0^t e^{i(t-t')\Delta}(|u|^2u)(t')dt'.$$

We will show that the map

$$A:u(t)\longmapsto e^{it\Delta}\phi_0+\Phi_u(t)$$

sends the space $\mathcal{C}([0,T)\times H^s(\mathbb{R}^3))\cap X$ into itself, and it is a contraction. First, using Strichartz inequality (2.9),

$$\|\Phi_u(t)\|_{L_t^qW_x^{s,p}([0,T)\times\mathbb{R}^3)}\lesssim\left\||u|^2u\right\|_{L_t^{8/5}W_x^{s,4/3}([0,T)\times\mathbb{R}^3)},$$

for all admissible pair (p,q). Now, by fractional Leibniz rule (A.19) and Sobolev inequality,

$$\begin{aligned}\||u|^2u\|_{L_t^{8/5}W_x^{s,4/3}([0,T)\times\mathbb{R}^3)} &\sim \|(\langle\nabla\rangle^s u)uu\|_{L_t^{8/5}L_x^{4/3}}\\ &\lesssim \|\langle\nabla\rangle^s u\|_{L_t^{8/3}L_x^4}\|u\|^2_{L_t^8L_x^4}\\ &\lesssim (2K_T)\|\langle\nabla\rangle^{1/2}u\|^2_{L_t^8L_x^{12/5}} \lesssim (2K_T)^3,\end{aligned}$$

Moreover, by Strichartz inequality (2.7), we also have

$$\begin{aligned}\sup_{[0,T]}\|\langle\nabla\rangle^s\Phi_u(t)\|_{L^2(\mathbb{R}^3)} &\lesssim \|\langle\nabla\rangle^s(|u|^2u)\|_{L_t^{8/5}L_x^{4/3}([0,T)\times\mathbb{R}^3)}\\ &\lesssim (2K_T)^3.\end{aligned}$$

Hence $\Phi_u(t)\in\mathcal{C}([0,T);H^s(\mathbb{R}^3))\cap X$ for T small enough and then the map A sends $\mathcal{C}([0,T);H^s(\mathbb{R}^3))\cap X$ itself for T sufficiently small, depending only on the size of ϕ_0 in $H^s(\mathbb{R}^3)$.

Now we'll show that A is a contraction. Let $u,v\in\mathcal{C}([0,T);H^s(\mathbb{R}^3))\cap X$. Using fractional Leibniz rule (A.19) and Sobolev embedding we have

$$\begin{aligned}\|\Phi_u(t)-\Phi_v(t)\|_X &= \|\int_0^t e^{i(t-t')\Delta}(|u|^2u-|v|^2v)dt'\|_X\\ &\lesssim \||u|^2u-|v|^2v\|_{L_t^{8/5}W_x^{s,4/3}}\\ &\le \|(u-v)\cdot(|u|^2+\overline{u}v)\|_{L_t^{8/5}W_x^{s,4/3}} + \|(\overline{u}-\overline{v})\cdot v^2\|_{L_t^{8/5}W_x^{s,4/3}}\\ &\lesssim \|\langle\nabla\rangle^s(u-v)\|_{L_t^{8/3}L_x^4}\||u|^2+\overline{u}v\|_{L_t^4L_x^2} + \|u-v\|_{L_t^8L_x^4}\|\langle\nabla\rangle^s(|u|^2+\overline{u}v)\|_{L_t^2L_x^2}\\ &\quad + \|\langle\nabla\rangle^s(\overline{u}-\overline{v})\|_{L_t^{8/3}L_x^4}\|v^2\|_{L_t^4L_x^2} + \|\overline{u}-\overline{v}\|_{L_t^8L_x^4}\|\langle\nabla\rangle^s(v^2)\|_{L_t^2L_x^2}\\ &\lesssim \left(\|\langle\nabla\rangle^s(u-v)\|_{L_t^{8/3}L_x^4} + \|\langle\nabla\rangle^{1/2}(u-v)\|_{L_t^8L_x^{12/5}}\right)\cdot(2K_T)^2\\ &\lesssim \|u-v\|_X\cdot(2K_T)^2.\end{aligned}$$

Analogously,

$$\sup_{[0,T)}\|\Phi_u(t)-\Phi_v(t)\|_{H^s(\mathbb{R}^3)} \lesssim \|u-v\|_X\cdot(2K_T)^2.$$

Then, for sufficiently small T we have that A is a contraction.

It remains to prove the continuous dependence of the initial data. Let ϕ be the solution associated to the initial data ϕ_0 considered below and $0<T'<T$. Write $\widetilde{\phi}$ for the solution associated to another initial data $\widetilde{\phi}_0$ close to ϕ_0 in $H^s(\mathbb{R}^3)$ and consider

$$\widetilde{K}_{T'} = \|e^{it\Delta}\widetilde{\phi}_0\|_{L_t^{8/3}W_x^{s,4}([0,T']\times\mathbb{R}^3)} + \|e^{it\Delta}\widetilde{\phi}_0\|_{L_t^8W_x^{s,12/5}([0,T']\times\mathbb{R}^3)}.$$

We have

$$|K_{T'}-\widetilde{K}_{T'}| \lesssim \|\phi_0-\widetilde{\phi}_0\|_{H^s(\mathbb{R}^3)}.$$

Then, if $\|\phi_0-\widetilde{\phi}_0\|_{H^s(\mathbb{R}^3)}$ is sufficiently small, we may use the argument employed below to conclude that

$$\|\phi-\widetilde{\phi}\|_X + \sup_{[0,T']}\|\phi-\widetilde{\phi}\|_{H^s(\mathbb{R}^3)} \lesssim \|\phi_0-\widetilde{\phi}_0\|_{H^s(\mathbb{R}^3)} + \|\phi-\widetilde{\phi}\|_X\left(K^2_{T'}+K_{T'}\widetilde{K}_{T'}+\widetilde{K}^2_{T'}\right)$$

and then

$$\|\phi - \widetilde{\phi}\|_X + \sup_{[0,T']} \|\phi - \widetilde{\phi}\|_{H^s(\mathbb{R}^3)} \leq C_{T'} \|\phi_0 - \widetilde{\phi}_0\|_{H^s(\mathbb{R}^3)}.$$

□

Remark 3.1. Note that this proof also is valid for $s = \frac{1}{2}$, which is the critical case. This coeficient preservs the $H^s(\mathbb{R}^n)$ norm of solutions by rescaling

$$\phi^{(\lambda)}(x,t) = \lambda \phi(\lambda x, \lambda^2 x),$$

and the estimated time of existence for this solutions depends on the profile of ϕ_0, and not only on its size. In our proof, we have not done this estimation, since we have just noted that $K_T \to 0$, but not how fast the limit yields. See [4] for a more detailed approx.

3.2. Conservation Laws

The next propositions show classical conservation laws for solutions of nonlinear Schrödinger equations.

Proposition 3.1. *Let u satisfy the integral equation* (3.2) *with $u_0 \in L^2(\mathbb{R}^n)$ and $u \in L^q([0,T); L^p(\mathbb{R}^n))$ for some admissible pair (p,q). Then*

$$\|u(t)\|_2 = \|u_0\|_2$$

for all $t \in [0,T)$.

Proof. Multiplying the integral equation (3.2) by $e^{-it\Delta}$ and taking L^2 norms we obtain

$$\begin{aligned}\|u(t)\|_2^2 &= \|e^{-it\Delta} u(t)\|_2^2 \\ &= \|u_0\|_2^2 + 2\mathscr{I}m \left\langle u_0, \int_0^t e^{-it'\Delta}(|u|^{\alpha-1}u)(t')dt' \right\rangle + \| \int_0^t e^{-it'\Delta}(|u|^{\alpha-1}u)(t')dt' \|_2^2.\end{aligned}$$

where we have taken $\lambda = 1$ for simplicity. We'll show that the sum of the last two terms on the right of the previous equation vanishes. The first of them is equal to

$$2\mathscr{I}m \int_0^t \langle e^{it'\Delta}u_0, |u|^{\alpha-1}u(t')\rangle dt'.$$

For the second one we write

$$\begin{aligned}\| \int_0^t e^{-it'\Delta}(|u|^{\alpha-1}u)(t')dt' \|_2^2 &= \int_0^t \int_0^t \langle e^{-it'\Delta}(|u|^{\alpha-1}u)(t'), e^{-it''\Delta}(|u|^{\alpha-1}u)(t'')\rangle dt'\, dt'' \\ &= \int_0^t \langle (|u|^{\alpha-1}u)(t'), \int_0^t e^{i(t'-t'')\Delta}(|u|^{\alpha-1}u)(t'')dt''\rangle dt' \\ &= 2\mathscr{R}e \int_0^t \langle (|u|^{\alpha-1}u)(t'), \int_0^{t'} e^{i(t'-t'')\Delta}(|u|^{\alpha-1}u)(t'')dt''\rangle dt' \\ &= 2\mathscr{I}m \int_0^t \langle (|u|^{\alpha-1}u)(t'), u(t') - i\int_0^{t'} e^{i(t'-t'')\Delta}(|u|^{\alpha-1}u)(t'')dt''\rangle dt' \\ &= 2\mathscr{I}m \int_0^t \langle (|u|^{\alpha-1}u)(t'), e^{it'\Delta}u_0\rangle dt' \\ &= -2\mathscr{I}m \int_0^t \langle e^{it'\Delta}u_0, |(u|^{\alpha-1}u)(t')\rangle dt'.\end{aligned}$$

Then $\|u(t)\|_2 = \|u_0\|_2$ for all $t \in [0,T)$.

□

Proposition 3.2. *Let $u_0 \in H^1(\mathbb{R}^n)$. Let $T > 0$ and let u be a solution of (3.2) with $u \in L^q([0,T); W^{1,p}(\mathbb{R}^n))$ for some admissible pair (p,q). Then*

$$E(u)(t) := \int_{\mathbb{R}^n} \left(|\nabla_x u(x,t)|^2 - \frac{2\lambda}{\alpha+1} |u(x,t)|^{\alpha+1} \right) dx = E(u_0).$$

Proof. In a way similar to the preceding argument (taking $\lambda = 1$), we compute

$$\begin{aligned}
&\|\nabla u(t)\|_2^2 \\
&\quad = \|\nabla e^{-it\Delta}(u(t))\|_2^2 \\
&\quad = \|\nabla u_0 - i \int_0^t e^{-it'\Delta}(\nabla(|u|^{\alpha-1}u)(t'))dt'\|_2^2 \\
&\quad = \|\nabla u_0\|_2^2 - 2\mathscr{I}m \left\langle \nabla u_0, \int_0^t e^{-it'\Delta}(\nabla(|u|^{\alpha-1}u)(t'))dt' \right\rangle \\
&\qquad + \| \int_0^t e^{-it'\Delta}(\nabla(|u|^{\alpha-1}u)(t'))dt'\|_2^2 \\
&\quad = \|u_0\|_2^2 - 2\mathscr{I}m \int_0^t \langle e^{it'\Delta}(\nabla u_0), \nabla(|u|^{\alpha-1}u)(t')dt' \\
&\qquad + 2\mathscr{R}e \int_0^t \langle (\nabla(|u|^{\alpha-1}u)(t')), \int_0^{t'} e^{i(t'-t'')\Delta}(\nabla(|u|^{\alpha-1}u)(t''))dt''\rangle dt' \\
&\quad = \|\nabla u_0\|_2^2 + 2\mathscr{I}m \int_0^t \langle \nabla(|u|^{\alpha-1}u)(t'), e^{it'\Delta}(\nabla u_0)\rangle dt' \\
&\qquad + 2\mathscr{I}m \int_0^t \langle \nabla(|u|^{\alpha-1}u)(t'), -i\int_0^{t'} e^{i(t'-t'')\Delta}(\nabla(|u|^{\alpha-1}u)(t''))dt''\rangle dt' \\
&\quad = \|\nabla u_0\|_2^2 + 2\mathscr{I}m \int_0^t \langle \nabla(|u|^{\alpha-1}u)(t'), \nabla u(t')\rangle dt' \\
&\quad = \|\nabla u_0\|_2^2 - 2\mathscr{I}m \int_0^t \langle (|u|^{\alpha-1}u)(t'), \Delta u(t')\rangle dt' \\
&\quad = \|\nabla u_0\|_2^2 - 4\mathscr{R}e \int_0^t \langle (|u|^{\alpha-1}u)(t'), \partial_t u(t')\rangle dt' \\
&\quad = \|\nabla u_0\|_2^2 - \frac{2}{\alpha+1} \int_0^t \frac{d}{dt} \int_{\mathbb{R}^n} |u(t')|^{\alpha+1} dx dt' \\
&\quad = \|\nabla u_0\|_2^2 - \frac{2}{\alpha+1} \int_{\mathbb{R}^n} |u(t)|^{\alpha+1} dx + \frac{2}{\alpha+1} \int_{\mathbb{R}^n} |u_0|^{\alpha+1} dx.
\end{aligned}$$

This completes the proof. A rigorous justification of identity

$$\mathscr{I}m\langle |u|^{\alpha-1}u, \Delta u\rangle = 2\mathscr{R}e\langle |u|^{\alpha-1}u, \partial_t u\rangle$$

may be seen in [14]. □

3.3. Global Theory

Now we show some examples of global well-posedness results and others of formation of singularities for nonlinear Schrödinger equations.

First consider the local solution of the IVP (3.1) with $u_0 \in H^1(\mathbb{R}^n)$ provided by Theorem 3.1. If $\lambda < 0$ (defocusing case), by conservation law 3.2 we have

$$\sup_{[0,T)} \int_{\mathbb{R}^n} |\nabla u(x,t)|^2 dx \leq E(u_0)$$

which combined with (3.1) gives

$$\sup_{[0,T)} \|u(t)\|^2_{H^1(\mathbb{R}^n)} \leq E(u_0) + \|u_0\|_2^2.$$

This allows us to extend the local solution u to any time interval.

In some other cases, local solutions can be extended to global solutions. The next result summarize some of them.

Theorem 3.4 ([13], page 122). *Under any of the following set of hypotheses the local solution of the IVP* (3.1) *with* $u_0 \in H^1(\mathbb{R}^n)$ *provided by Theorem* 3.1 *extends globally in time.*

1. $\lambda < 0$,
2. $\lambda > 0$ *and* $\alpha < 1 + 4/n$,
3. $\lambda > 0$, $\alpha = 1 + 4/n$, *and* $\|u_0\|_2 < c_0$, *for some constant* c_0 *depending on* u_0.
4. $\lambda > 0$, $\alpha > 1 + 4/n$, *and* $\|u_0\|_{H^1(\mathbb{R}^n)} \leq \rho$, *for* ρ *sufficiently small.*

This theorem is optimal. Now we prove that, if *4.* does not hold, then there exists $u_0 \in H^1(\mathbb{R}^n)$ and $T^* < \infty$ such that the corresponding solution u of the IVP (3.1) satisfies

$$\lim_{t \uparrow T^*} \|\nabla u(t)\|_2 = \infty. \tag{3.9}$$

To simplify the exposition we shall assume $\lambda = 1$. In the proof of this result we need the following identities, whose proof is easy but cumbersome and will be omited here.

Proposition 3.3 ([13], page 124). *If* $u(t)$ *is a solution in* $\mathcal{C}([0,T); H^1(\mathbb{R}^n))$ *of the IVP* (3.1) *with* $\lambda = 1$ *obtained in Theorem* 3.1, *then*

$$\frac{d}{dt} \int_{\mathbb{R}^n} |x|^2 |u(x,t)|^2 dx = 4 \mathscr{I}m \int_{\mathbb{R}^n} r \overline{u} \partial_r u dx, \tag{3.10}$$

with $r = |x|$, *and*

$$\frac{d}{dt} \mathscr{I}m \int_{\mathbb{R}^n} r \overline{u} \partial_r u dx = 2 \int_{\mathbb{R}^n} |\nabla u(x,t)|^2 dx + \left(\frac{2n}{\alpha + 1} - n \right) \int_{\mathbb{R}^n} |u(x,t)|^{\alpha+1} dx. \tag{3.11}$$

The next result is implicity needed to give sense of the above identities.

Proposition 3.4 ([13], page 126)**.** *If $u(t)$ is a solution in $\mathcal{C}([0,T);H^1(\mathbb{R}^n))$ of the IVP* (3.1) *with $\lambda = 1$ provided by Theorem* 3.1 *such that $x_j u_0 \in L^2(\mathbb{R}^n)$ for some $j = 1, \dots, n$, then*

$$x_j u(\cdot,t) \in \mathcal{C}([0,T);L^2(\mathbb{R}^n)).$$

Thus, if $u_0 \in L^2(\mathbb{R}^n, |x|^2 dx)$, then

$$u(\cdot,t) \in \mathcal{C}([0,T);H^1 \cap L^2(|x|^2 dx)).$$

Theorem 3.5 ([13], page 126)**.** *Let u be a solution in $\mathcal{C}([0,T);H^1(\mathbb{R}^n) \cap L^2(|x|^2 dx))$ of the IVP* (3.1) *with $\lambda = 1$ provided by Theorem* 3.1 *and Proposition* 3.4*. Assume that the initial data u_0 and the nonlinearity α satisfy the following assumptions:*

1. $\displaystyle\int_{\mathbb{R}^n} \left(|\nabla u_0|^2 - \frac{2}{\alpha+1}|u_0|^{\alpha+1} \right) dx = E(u_0) = E_0 < 0,$
2. $\alpha \in (1 + 4/n, 1 + 4/(n-2))$;

then there exists $T^* > 0$ such that

$$\lim_{t \uparrow T^*} \|\nabla u(t)\|_2 = \infty.$$

We observe that condition *1.* implies that $\|u_0\|_{H^1(\mathbb{R}^n)}$ is not arbitrarily small. In particular, for any $u_0 \in H^1(\mathbb{R}^n)$ one has that $E_0(\gamma u_0) < 0$ for $\gamma > 0$ sufficiently large.

Proof. We first assume that $\mathscr{I}m \left(\int_{\mathbb{R}^n} r\overline{u}_0 \partial_r u_0 dx \right) < 0$. We define the function

$$f(t) = -\mathscr{I}m \int_{\mathbb{R}^n} r(\partial_r u \overline{u})(x,t) dx.$$

By our assumption $f(0) > 0$. Using identity (3.11) and the definition of E_0 it follows that

$$\begin{aligned}
f'(t) &= -2\int_{\mathbb{R}^n} |\nabla u(x,t)|^2 dx - \left(\frac{2n}{\alpha+1} - n \right) \int_{\mathbb{R}^n} |u(x,t)|^{\alpha+1} dx \\
&= -2\int_{\mathbb{R}^n} |\nabla u(x,t)|^2 dx + n\left(\frac{\alpha+1}{2} - 1 \right) \frac{2}{\alpha+1} \int_{\mathbb{R}^n} |u(x,t)|^{\alpha+1} dx \\
&= -2\int_{\mathbb{R}^n} |\nabla u(x,t)|^2 dx + n\left(\frac{\alpha+1}{2} - 1 \right) \left(\int_{\mathbb{R}^n} |\nabla u(x,t)|^2 dx - E_0 \right) \\
&= -\left[2 - n\left(\frac{\alpha+1}{2} - 1 \right) \right] \int_{\mathbb{R}^n} |\nabla u(x,t)|^2 dx - n\left(\frac{\alpha+1}{2} - 1 \right) E_0 \\
&\geq M \|\nabla u(t)\|_2^2,
\end{aligned}$$

since by hypothesis $E_0 < 0$, $\alpha > 1$ implies that $(\alpha+1)/2 - 1 > 0$, and $\alpha > 1 + 4/n$ implies that $n((\alpha+1)/2 - 1) - 2 = M > 0$. Then $f(t)$ is an increasing function, so $f(t) \geq f(0) > 0$ for all $t > 0$.

Now we use (3.10) to see that

$$\frac{d}{dt}\int_{\mathbb{R}^n} |x|^2 |u(x,t)|^2 dx = 4\mathscr{I}m \int_{\mathbb{R}^n} r(\overline{u}\partial_r u)(x,t) dx = -4f(t) < 0.$$

Thus, $h(t) = \int |x|^2 |u(x,t)|^2 dx$ is a decreasing function with

$$h(t) \leq \int_{\mathbb{R}^n} |x|^2 |u_0(x)|^2 dx = h(0)$$

Applying Cauchy-Schwartz inequality we obtain

$$\begin{aligned}
|f(t)| = f(t) &= -\mathscr{I}m \int_{\mathbb{R}^n} r(\overline{u}\partial_r u)(x,t)dx \\
&\leq \left(\int_{\mathbb{R}^n} r^2 |u|^2 (x,t) dx \right)^{1/2} \left(\int_{\mathbb{R}^n} |\partial_r u|^2 (x,t) dx \right)^{1/2} \\
&\leq (h(0))^{1/2} \|\nabla u(t)\|_2,
\end{aligned}$$

and then $f(t)$ satisfies the differential inequality

$$\begin{cases} f'(t) \geq \dfrac{M}{h(0)} (f(t))^2, \\ f(0) > 0. \end{cases}$$

Hence,

$$(h(0))^{1/2} \|\nabla u(t)\|_2 \geq f(t) \geq \frac{h(0) f(0)}{h(0) - M f(0) t}.$$

Defining

$$T_0 = \frac{h(0)}{M f(0)} > 0$$

we obtain that

$$\lim_{t \uparrow T^*} \|\nabla u(t)\|_2 = \infty$$

with $T^* = T_0$.

It remains to prove the result in the case $\mathscr{I}m \left(\int r \overline{u}_0 \partial_r u_0 dx \right) \geq 0$. In this case we have

$$\frac{d}{dt} \mathscr{I}m \int_{\mathbb{R}^n} r \overline{u} \partial_r u(x,t) dx = 2E_0 + \left(\frac{2(n+2)}{\alpha+1} - n \right) \int_{\mathbb{R}^n} |u(x,t)|^{\alpha+1} dx \leq 2E_0$$

because $\alpha > 1 + 4/n$. Hence since $E_0 < 0$ there exists $\hat{t} > 0$ such that

$$\mathscr{I}m \left(\int_{\mathbb{R}^n} r \overline{u} \partial_r u(x, \hat{t}) dx \right) < 0$$

and we are in the case previously considered. $\square$

§4. The Cubic NLS Equation on $\mathbb{R}^3$

In this section we prove the main result of the work. We prove global existence and scattering for the solutions of the defocusing, cubic, nonlinear Schrödinger equation in $H^s(\mathbb{R}^3)$ for $s > \frac{4}{5}$, giving a detailed version of [7].

We consider the following initial value problem,

$$i\partial_t \phi(x,t) + \Delta\phi(x,t) = |\phi(x,t)|^2\phi(x,t), \quad x \in \mathbb{R}^3,\ t \geq 0, \tag{4.1}$$

$$\phi(x,0) = \phi_0(x) \in H^s(\mathbb{R}^3). \tag{4.2}$$

We have proven in theorem 3.3 of the previous section that (4.1)-(4.2) is well-posed locally in time in $H^s(\mathbb{R}^3)$ when $s > \frac{1}{2}$. In this case, the *Schrödinger admissible exponents* for $\mathbb{R}^{3+1}$ when $q, r \geq 2$ are

$$\frac{1}{q} + \frac{3}{2r} = \frac{3}{4}. \tag{4.3}$$

In addition, these local solutions satisfy the L^2 conservation

$$\|\phi(\cdot,t)\|_{L^2(\mathbb{R}^3)} = \|\phi_0(\cdot)\|_{L^2(\mathbb{R}^3)}, \tag{4.4}$$

and the $H^1(\mathbb{R}^3)$ solutions have the following conserved energy,

$$E(\phi)(t) = \int_{\mathbb{R}^3} \frac{1}{2}|\nabla_x\phi(x,t)|^2 + \frac{1}{4}|\phi(x,t)|^4 dx = E(\phi)(0). \tag{4.5}$$

As we have also seen, these conservations laws and the local-in-time theory implies the global-in-time well-posedness for data in H^s when $s \geq 1$. However, when $s < 1$, the energy of our solutions is generally infinity, so we can not use the conservation law to prove global existence. Our aim will be to control the growth of $E(I\phi)(t)$ uniformly in time, where $I\phi$ will be a smothed version of ϕ. That is what we will call an *almost conservation law*. Moreover, for our arguments, we need to include an estimate based in the classical *Morawetz action*. While the *standard Morawetz-type estimate* bounds

$$\int_0^\infty \int_{\mathbb{R}^3} \frac{(\phi(x,t))^4}{|x|} dx\, dt,$$

we introduce an interaction potential generalization of the Morawetz action, called *Morawetz Interaction Potential*, to control

$$\|\phi(x,t)\|_{L^4_{x,t}(\mathbb{R}^3\times\mathbb{R})}.$$

Finally, we write $S^{\mathrm{L}}(t)$ for the flow map $e^{it\Delta}$ corresponding to the linear Schrödinger equation, and $S^{\mathrm{NL}}(t)$ for the nonlinear flow, that is, $S^{\mathrm{NL}}(t)\phi_0 = \phi(x,t)$, with ϕ and ϕ_0 as in (4.1)-(4.2). Given a solution $\phi \in \mathcal{C}((-\infty,\infty), H^s(\mathbb{R}^3))$ of (4.1)-(4.2), we define the asymptotic states $\phi^{\pm}$ and wave operators $\Omega^{\pm} : H^s(\mathbb{R}^3) \to H^s(\mathbb{R}^3)$ by

$$\phi^{\pm} = \lim_{t\to\pm\infty} S^{\mathrm{L}}(-t) S^{\mathrm{NL}}(t)\phi_0 \tag{4.6}$$

$$\Omega^{\pm}\phi^{\pm} = \phi_0, \tag{4.7}$$

insofar as these limits exist in $H^s(\mathbb{R}^3)$. When the wave operators $\Omega^{\pm}$ are surjective, we say that (4.1)-(4.2) is *asymptotically complete* in $H^s(\mathbb{R}^3)$.

Our main result is the following:

Theorem 4.1. *The IVP* (4.1)-(4.2) *is globally well-posed from data* $\phi_0 \in H^s(\mathbb{R}^3)$ *when* $s > 4/5$. *In addition, the wave operators* (4.7) *exist and there is asymptotic completeness on all of* $H^s(\mathbb{R}^3)$.

4.1. Morawetz Interaction Potential

The discussion here will be carried out in the context of the more general form of (4.1) given by

$$i\partial_t u + \alpha\Delta u = \mu f(|u|^2)u, \quad u : \mathbb{R}\times\mathbb{R}^3 \to \mathbb{C}, \tag{4.8}$$

$$u(0) = u_0, \tag{4.9}$$

where f is a smooth function $f : \mathbb{R}^+ \to \mathbb{R}^+$ and α and μ are real constants. We define $F(z) = \int_0^z f(s)$.

We will also use spherical coordinates $r\omega$, $r > 0$, $\omega \in \mathbb{S}^2$. In this context, we use the gradient on the sphere, $\nabla_\omega : \mathcal{C}^\infty(\mathbb{S}^2) \to T\mathbb{S}^2$, defined by

$$\langle \nabla_\omega g(x), v\rangle = D_x g(v), \quad \forall v \in T_x\mathbb{S}^2, \ \forall x \in \mathbb{S}^2.$$

and its extension to $\mathbb{R}^3 \setminus \{0\}$, which is given by

$$\nabla_\omega f := r\nabla_0 f, \tag{4.10}$$

where $\nabla_0 = \nabla_{r\omega}$ is the angular component of the derivative of f (or the gradient on the sphere $r\mathbb{S}^2$), that is to say

$$\nabla_0 f = \nabla f - \left\langle \frac{x}{|x|}, \nabla f \right\rangle \frac{x}{|x|}, \tag{4.11}$$

or, writen in spherical coordinates,

$$\nabla f = \nabla_{r\omega} f + \partial_r f \frac{\partial}{\partial r}. \tag{4.12}$$

Moreover, we define the Laplace-Beltrami operator on the sphere by $\Delta_\omega = \operatorname{div}\nabla_\omega$, and recall that the divergence of a vector field X over $\mathbb{S}^2$ is given by

$$\langle -\mathrm{div} X, g\rangle_{\mathbb{S}^2} = \langle X, \nabla_\omega g\rangle_{\mathbb{S}^2} := \int_{\mathbb{S}^2} \langle X, \nabla_\omega g\rangle dS,$$

where $\langle \cdot\,, \cdot\rangle_{\mathbb{S}^2}$ denotes the inner product in $L^2(\mathbb{S}^2)$. The extension of Δ_ω to $\mathbb{R}^3 \setminus \{0\}$ is

$$\Delta_\omega f = r^2 \Delta_{r\omega} f,$$

where, as in the case of the gradient, $\Delta_{r\omega}$ denotes the Laplace-Beltrami operator of the sphere of radius r. From (4.10) and (4.12), it follows that the relation between the spherical Laplacian and the usual one is

$$\Delta f = \frac{1}{r^2}\Delta_\omega f + \frac{2}{r}\frac{\partial f}{\partial r} + \frac{\partial^2 f}{\partial r^2}.$$

Now, we recall some alternate forms of the equation (4.8):

$$u_t = i\alpha\Delta u - i\mu f((|u|^2)u, \tag{4.13}$$

$$\overline{u}_t = -i\alpha\Delta\overline{u} + i\mu f(|u|^2)\overline{u}, \tag{4.14}$$

$$u_t = i\alpha u_{rr} + i\frac{2\alpha}{r}u_r + i\frac{\alpha}{r^2}\Delta_\omega u - i\mu f(|u|^2)u, \tag{4.15}$$

$$(ru_t) = i\alpha(ru)_{rr} + i\frac{\alpha}{r}\Delta_\omega u - i\mu f(|u|^2)u, \tag{4.16}$$

$$(r\overline{u}_t) = -i\alpha(r\overline{u})_{rr} - i\frac{\alpha}{r}\Delta_\omega\overline{u} + i\mu f(|u|^2)\overline{u}. \tag{4.17}$$

We introduce the *Morawetz action centered at* 0 for the solution u of (4.8),

$$M_0[u](t) = \int_{\mathbb{R}^3}\left\langle \mathscr{I}m[\overline{u}(t,x)\nabla u(t,x)], \frac{x}{|x|}\right\rangle dx. \tag{4.18}$$

We can verify, using (4.13) and (4.14) that

$$\partial_t(|u|^2) = i\alpha(\overline{u}\Delta u - u\Delta\overline{u}) = -2\alpha\langle\nabla, \mathscr{I}m[\overline{u}(t,x)\nabla u(t,x)]\rangle, \tag{4.19}$$

so we may interpret M_0 as the spatial average of the radial component of the L^2 mass current. In order to obtain global existence, it is spected that the solution scatters with time, so M_0 should increase, since that behavior involves a redistribution of the L^2 mass, which is what we mean by dispersion. The following result stablishes that $\frac{d}{dt}M_0[u](t) \geq 0$ for defocusing equations.

Proposition 4.1. *If u solves* (4.8)-(4.9)*, then the Morawetz action at* 0 *satisfies the identity*

$$\begin{aligned}\partial_t M_0[u](t) = 4\pi\alpha|u(t,0)|^2 &+ \int_{\mathbb{R}^3}\frac{2\alpha}{|x|}|\nabla_0 u(t,x)|^2 dx \\ &+ \mu\int_{\mathbb{R}^3}\frac{2}{|x|}\{|u|^2 f(|u|^2)(t) - F(|u|^2)(t)\}dx.\end{aligned} \tag{4.20}$$

In particular, M_0 is an increasing function of time if the equation (4.8) *satisfies the repulsion condition,*

$$\mu\{|u|^2 f(|u|^2)(t) - F(|u|^2)(t)\} \geq 0. \tag{4.21}$$

Remark 4.1. For pure power potentials

$$F(x) = \frac{2}{p+1}x^{\frac{p+1}{2}},$$

where the nonlinear term in (4.8) is $|u|^{p-1}u$, we have

$$|u|^2 f(|u|^2) - F(|u|^2) = |u|^{p+1}\left(1 - \frac{2}{p+1}\right) = \frac{p-1}{2}F(|u|^2).$$

Hence condition (4.21) holds in the defocusing case $\mu \geq 0$.

Proof. Since $\langle \nabla u, \frac{x}{|x|}\rangle = \partial_r u$ and $|u|^2 = u\overline{u}$ is real, we may write

$$M_0(t) = \mathscr{I}m \int_{\mathbb{R}^3} \overline{u}(t,x)\left(\partial_r + \frac{1}{r}\right) u(t,x)\, dx \tag{4.22}$$

$$= \mathscr{I}m \int_0^\infty \int_{\mathbb{S}^2} \overline{ru}(ru)_r \, d\omega\, dr, \tag{4.23}$$

Integrating by parts and using the equation (4.16) gives

$$\begin{aligned}
\frac{d}{dt}M_0 =& \mathscr{I}m \int_0^\infty \int_{\mathbb{S}^2} \overline{(ru)}(ru_t)_r + \overline{(ru_t)}(ru)_r \, d\omega\, dr \\
=& \mathscr{I}m \int_0^\infty \int_{\mathbb{S}^2} -\overline{(ru)_r}(ru_t) + \overline{(ru_t)}(ru)_r \, d\omega\, dr \\
=& -2\mathscr{I}m \int_0^\infty \int_{\mathbb{S}^2} \overline{(ru)_r}(ru_t)\, d\omega\, dr \\
=& -2\mathscr{I}m \int_0^\infty \int_{\mathbb{S}^2} \overline{(ru)_r}\left\{ i\alpha(ru)_{rr} + i\frac{\alpha}{r}\Delta_\omega u - i\mu r f(|u|^2)u\right\} d\omega\, dr \\
=& -2\alpha\,\mathscr{R}e \int_0^\infty \int_{\mathbb{S}^2} \overline{(ru)_r}(ru)_{rr} d\omega\, dr - 2\alpha\,\mathscr{R}e \int_0^\infty \int_{\mathbb{S}^2} \overline{(ru)_r}\frac{1}{r}\Delta_\omega u\, d\omega\, dr \\
&+ 2\mu\,\mathscr{R}e \int_0^\infty \int_{\mathbb{S}^2} \overline{(ru)_r} r f(|u|^2)u\, d\omega\, dr \\
=& \mathrm{I} + \mathrm{II} + \mathrm{III}.
\end{aligned}$$

We are going to see separately that these three terms lead to the three terms on the right side of (4.20), respectively.

Term I. Since $\partial_r |(ru)_r|^2 = 2\mathscr{R}e\overline{(ru)_r}(ru)_{rr}$, we obtain

$$\begin{aligned}
\mathrm{I} &= -\alpha \int_{\mathbb{S}^2} \left[|(ru)_r|^2\right]_0^\infty d\omega = -\alpha \int_{\mathbb{S}^2} \left[(ru_r + u)\overline{(ru_r + u)}\right]_0^\infty d\omega \\
&= \alpha \int_{\mathbb{S}^2} |u(t,0)|^2 d\omega = 4\pi\alpha |u(t,0)|^2.
\end{aligned}$$

Term II. Recall that $\Delta_\omega = \operatorname{div}\nabla_\omega$, so we can write

$$\begin{aligned}
\mathrm{II} &= 2\alpha\,\mathscr{R}e \int_0^\infty \int_{\mathbb{S}^2} \left\langle \nabla_\omega\left(\frac{1}{r}\overline{u} + \overline{u_r}\right), \nabla_\omega u\right\rangle d\omega\, dr \\
&= \alpha\,\mathscr{R}e \int_0^\infty \int_{\mathbb{S}^2} \left[\frac{2}{r}|\nabla_\omega u|^2 + \partial_r |\nabla_\omega u|^2\right] d\omega\, dr.
\end{aligned}$$

Now, since $\nabla_\omega u = r\nabla u - ru_r\frac{\partial}{\partial r}$, we know that $|\nabla_\omega u|$ vanishes at the origin and then the second term integrates to 0. Therefore, we can write term II as

$$\begin{aligned}
\mathrm{II} &= \alpha \int_0^\infty \int_{\mathbb{S}^2} \frac{r^2}{r^2}\frac{2}{r}|\nabla_\omega|^2 d\omega\, dr \\
&= \alpha \int_0^\infty \int_{\mathbb{S}^2} r^2\frac{2}{r}|\nabla_0 u|^2 d\omega\, dr \\
&= \int_{\mathbb{R}^3} \frac{2\alpha}{|x|}|\nabla_0 u|^2 dx.
\end{aligned}$$

Term III. Here we use the expansion

$$(\overline{u} + r\overline{u_r})rf(|u|^2)u = r|u|^2 f(|u|^2) + r^2 f(|u|^2)u\overline{u_r}.$$

The first of these terms is purely real-valued. The real part of the second term may be writen as

$$\mathscr{Re}\, r^2 f(|u|^2)u\overline{u_r} = \frac{1}{2}\left[F(|u|^2)_r\right].$$

Then, integrating by parts with respect to r, and changing to cartesian coordinates once again, conclude that

$$\begin{aligned}
\mathrm{III} &= 2\mu \int_0^\infty \int_{\mathbb{S}^2} r|u|^2 f(|u|^2) + \frac{r^2}{2}\left[F(|u|^2)\right]_r d\omega\, dr \\
&= \mu \int_0^\infty \int_{\mathbb{S}^2} 2r|u|^2 f(|u|^2) - 2rF(|u|^2) d\omega\, dr \\
&= \mu \int_{\mathbb{R}^3} \frac{2}{|x|}\left\{|u|^2 f(|u|^2) - F(|u|^2)\right\} dx.
\end{aligned}$$

□

We may repeat the above argument centering it at any other point $y \in \mathbb{R}^3$, defining the *Morawetz action at* y to be

$$M_y[u](t) = \int_{\mathbb{R}^3} \left\langle \mathscr{Im}[\overline{u}(x)\nabla u(x)], \frac{x-y}{|x-y|}\right\rangle dx. \tag{4.24}$$

Corollary 4.3. *If u solves* (4.8)-(4.9)*, then the Morawetz action at y satisfies the identity*

$$\begin{aligned}
\partial_t M_y[u](t) &= 4\pi\alpha|u(t,y)|^2 + \int_{\mathbb{R}^3} \frac{2\alpha}{|x-y|}|\nabla_y u(t,x)|^2 dx \\
&\quad + \mu \int_{\mathbb{R}^3} \frac{2}{|x-y|}\{|u|^2 f(|u|^2)(t) - F(|u|^2)(t)\} dx,
\end{aligned} \tag{4.25}$$

where

$$\nabla_y u = \nabla u - \frac{x-y}{|x-y|}\left\langle \frac{x-y}{|x-y|}, \nabla u\right\rangle.$$

In particular, M_y is an increasing function of time if the equation (4.8) *satisfies the repulsion condition* (4.21).

For our scattering results, we will need the following pointwise bound for $M_y[u](t)$:

Lemma 4.2. *Assume u is a solution of* (4.8) *and $M_y[u](t)$ as in* (4.24)*. Then,*

$$|M_y(t)| \lesssim \|u(t)\|^2_{\dot{H}^{1/2}_x(\mathbb{R}^3)}. \tag{4.26}$$

Proof. Assume, without loss of generality, that $y = 0$. Then, by duality

$$\left| \mathscr{I}m \int_{\mathbb{R}^3} \overline{u(x,t)} \partial_r u(x,t) dx \right| \leq \|u\|_{\dot{H}^{1/2}(\mathbb{R}^3)} \cdot \|\partial_r u\|_{\dot{H}^{-1/2}(\mathbb{R}^3)}.$$

It suffices to show $\|\partial_r u\|_{\dot{H}^{-1/2}(\mathbb{R}^3)} \lesssim \|u\|_{\dot{H}^{1/2}(\mathbb{R}^3)}$. Since $\partial_r = \left\langle \frac{x}{|x|}, \nabla \right\rangle$, we have

$$\begin{aligned} \left\| \left\langle \frac{x}{|x|}, \nabla u \right\rangle \right\|_{\dot{H}^{1/2}} &= \sup_{\|f\|_{\dot{H}^{1/2}}=1} \int_{\mathbb{R}^3} \left\langle \frac{x}{|x|}, \nabla u \right\rangle f(x) dx \\ &\leq \sup_{\|f\|_{\dot{H}^{1/2}}=1} \|\nabla u\|_{\dot{H}^{-1/2}(\mathbb{R}^3)} \cdot \left\| \frac{x}{|x|} f \right\|_{\dot{H}^{1/2}(\mathbb{R}^3)} \\ &\leq \sup_{\|f\|_{\dot{H}^{1/2}}=1} \|u\|_{\dot{H}^{1/2}(\mathbb{R}^3)} \cdot \left\| \frac{x}{|x|} f \right\|_{\dot{H}^{1/2}(\mathbb{R}^3)}. \end{aligned}$$

Then, it remains to prove

$$\left\| \frac{x}{|x|} f \right\|_{\dot{H}^{1/2}(\mathbb{R}^3)} \lesssim \|f\|_{\dot{H}^{1/2}(\mathbb{R}^3)}.$$

for any f for which the right side is finite. By interpolating between L^2 and H^1, we need to show

$$\left\| \frac{x}{|x|} f \right\|_{L^2(\mathbb{R}^3)} \leq \|f\|_{L^2(\mathbb{R}^3)}, \qquad \left\| \frac{x}{|x|} f \right\|_{\dot{H}^1(\mathbb{R}^3)} \lesssim \|f\|_{\dot{H}^1(\mathbb{R}^3)}.$$

The first of these two bounds is trivial. For proving the second, we use the Hardy's inequality (A.18),

$$\left\| \nabla \left(\frac{x}{|x|} f \right) \right\|_{L^2} \leq \left\| \left\langle \frac{x}{|x|}, \nabla f \right\rangle \right\|_{L^2} + \left\| \frac{1}{|x|} f \right\|_{L^2} \lesssim \|\nabla f\|_{L^2}.$$

□

Corollary 4.4 (Morawetz Inequalities)**.** *Suppose u is a solution of* (4.8)-(4.9)*. Then for any $y \in \mathbb{R}^3$,*

$$\begin{aligned} 2 \sup_{t\in[0,T]} \|u(t)\|^2_{\dot{H}^{1/2}_x} &\gtrsim 4\pi\alpha \int_0^T |u(t,y)|^2 dt \\ &\quad + \int_0^T \int_{\mathbb{R}^3} \frac{2\alpha}{|x-y|} |\nabla_y u(t,x)|^2 dx\, dt \\ &\quad + \int_0^T \int_{\mathbb{R}^3} \frac{2\mu}{|x-y|} \{|u|^2 f(|u|^2) - F(|u|^2)\} dx\, dt. \end{aligned} \tag{4.27}$$

In particular, the inequality gives a bound uniform in T for the quantity

$$\int_0^T \int_{\mathbb{R}^3} \frac{|u(t,x)|^4}{|x-y|} dx\, dt$$

for solutions of (4.1)-(4.2)*.*

Now, we introduce a generalization of the Morawetz action in order to obtain a bound uniform in time for the $L^4([0,T]\times\mathbb{R}^3)$ norm of our solutions of (4.1)-(4.2).

Definition 4.1. *Given a solution u of* (4.8)*, the Morawetz interaction potential is defined by*

$$M[u](t) = \int_{\mathbb{R}^3} |u(t,y)|^2 M_y[u](t)\,dy, \tag{4.28}$$

that is, the average of $M_y[u](t)$ with respect to the density $|u(t,y)|^2$. We'll drop u in the notation below, writing $M[u](t) = M(t)$.

The bound (4.26) implies that

$$|M(t)| \lesssim \|u(t)\|_{L^2}^2 \|u(t)\|_{\dot H^{1/2}}^2. \tag{4.29}$$

Using the identity (4.25), we obtain the following one:

$$\begin{aligned}\frac{d}{dt}M(t) = {}& 4\pi\alpha\int_{\mathbb{R}^3}|u(y)|^4dy + \int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\frac{2\alpha}{|x-y|}|u(y)|^2|\nabla_y u(x)|^2dx\,dy\\ &+\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\frac{2\mu}{|x-y|}|u(y)|^2\{|u(x)|^2f(|u(x)|^2)-F(|u(x)|^2)\}dx\,dy\\ &+\int_{\mathbb{R}^3}\partial_t(|u(t,y)|^2)M_y(t)dy.\end{aligned}\tag{4.30}$$

We write the right hand side of (4.30) as $\mathrm{I}+\mathrm{II}+\mathrm{III}+\mathrm{IV}$, and we are going to see that this sum can be rewriten as a sum involving only nonnegative terms.

Proposition 4.2. *Referring to the terms of* (4.30)*, we have*

$$\mathrm{IV} \geq -\mathrm{II}. \tag{4.31}$$

As a consequence, we have that solutions of (4.8) *satisfy*

$$\begin{aligned}\frac{d}{dt}M(t) \geq {}& 4\pi\alpha\int_{\mathbb{R}^3}|u(y)|^4dy\\ &+\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\frac{2\mu}{|x-y|}|u(y)|^2\{|u(x)|^2f(|u(x)|^2)-F(|u(x)|^2)\}dx\,dy.\end{aligned}\tag{4.32}$$

In particular, $M(t)$ is monotone increasing for equations involving the repulsion condition (4.21)*.*

Proof. Using (4.19), write

$$\begin{aligned}\mathrm{IV} = {}& -2\alpha\int_{\mathbb{R}^3}\langle\nabla, \mathscr{I}m[\overline{u}(t,y)\nabla u(t,y)]\rangle M_y(t)\,dy\\ = {}& -2\alpha\int_{\mathbb{R}^3_y}\int_{\mathbb{R}^3_x}\left(\sum_{l=1}^3\partial_{y_l}\mathscr{I}m[\overline{u}(y)\partial_{y_l}u(y)]\right)\cdot\left(\sum_{m=1}^3\mathscr{I}m\left[\overline{u}(x)\frac{x_m-y_m}{|x-y|}\partial_{x_m}u(x)\right]\right)dxdy\\ = {}& 2\alpha\int_{\mathbb{R}^3_y}\int_{\mathbb{R}^3_x}\sum_{l=1}^3\left\{\mathscr{I}m[\overline{u}(y)\partial_{y_l}u(y)]\cdot\sum_{m=1}^3\mathscr{I}m\left[\overline{u}(x)\partial_{y_l}\left(\frac{x_m-y_m}{|x-y|}\right)\partial_{x_m}u(x)\right]\right\}dxdy,\end{aligned}$$

where we have integrated by parts in the last equality, moving the leading ∂_{y_l} to the unit vector $\frac{x-y}{|x-y|}$. Note that

$$\partial_{y_l}\left(\frac{x_m-y_m}{|x-y|}\right)=\frac{-\delta_{lm}}{|x-y|}+\frac{(x_l-y_l)(x_m-y_m)}{|x-y|^3}.$$

Using this identity and the notation $\mathbf{p}(x)=\mathscr{I}m[\overline{u}(x)\nabla u(x)]$ for the mass current at x, we obtain

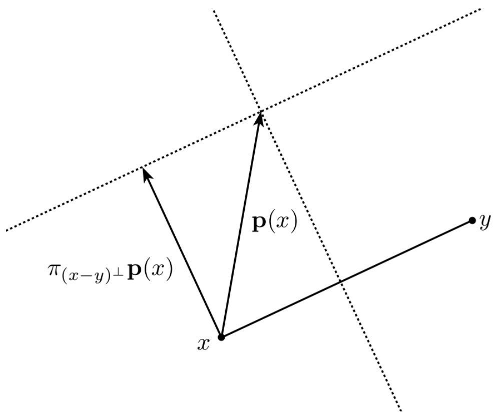
$$\mathrm{IV}=-2\alpha\int_{\mathbb{R}^3_y}\int_{\mathbb{R}^3_x}\left[\langle\mathbf{p}(y),\mathbf{p}(x)\rangle-\left\langle\mathbf{p}(y),\frac{x-y}{|x-y|}\right\rangle\left\langle\mathbf{p}(x),\frac{x-y}{|x-y|}\right\rangle\right]\frac{dxdy}{|x-y|}. \tag{4.33}$$

The geometric interpretation of the preceding integrand is now clear. We are removing the components of the inner product of $\mathbf{p}(x)$ and $\mathbf{p}(y)$ parallel to the vector $\frac{x-y}{|x-y|}$, so we can rewrite the previous integral as

$$-2\alpha\int_{\mathbb{R}^3_y}\int_{\mathbb{R}^3_x}\left\langle\pi_{(x-y)^\perp}\mathbf{p}(y),\pi_{(x-y)^\perp}\mathbf{p}(x)\right\rangle\frac{dxdy}{|x-y|},$$

where

$$\pi_{(x-y)^\perp}\mathbf{p}(\cdot):=\mathbf{p}(\cdot)-\frac{x-y}{|x-y|}\left\langle\frac{x-y}{|x-y|},\mathbf{p}(\cdot)\right\rangle.$$



But now, we have

$$|\pi_{(x-y)^\perp}\mathbf{p}(y)|=|\mathscr{I}m[\overline{u}(y)\nabla_x u(y)]|\leq|u(y)||\nabla_x u(y)|, \tag{4.34}$$

and the similar inequality switching the roles of x and y in (4.34). Hence,

$$\begin{aligned}\mathrm{IV}&\geq-2\alpha\int_{\mathbb{R}^3_y}\int_{\mathbb{R}^3_x}|u(x)||\nabla_y u(x)||u(y)|\nabla_x u(y)|\frac{dxdy}{|x-y|}\\&\geq-2\alpha\int_{\mathbb{R}^3_y}\int_{\mathbb{R}^3_x}|u(y)|^2|\nabla_y u(x)|^2\frac{dxdy}{|x-y|}=-\mathrm{II},\end{aligned}$$

where the last inequality follows by applying the elementary bound $|ab|\leq\frac{1}{2}(a^2+b^2)$ with $a=|u(y)|\cdot|\nabla_y u(x)|$ and $b=|u(x)|\cdot|\nabla_x u(y)|$. □

We combine (4.29) and (4.30) to obtain the following estimate.

Corollary 4.5. *Take u to be a smooth solution to the (IVP)* (4.8)-(4.9) *above under the repulsion condition* (4.21)*. Then we have the following Morawetz inequalities:*

$$
\begin{aligned}
2\|u(0)\|_{L^2}^2 \sup_{t\in[0,T]} \|u(t)\|^2_{\dot{H}_x^{1/2}} &\gtrsim 4\pi\alpha \int_0^T \int_{\mathbb{R}^3} |u(t,y)|^4 dy\, dt \\
&+ \int_0^T \int_{\mathbb{R}^3_y} \int_{\mathbb{R}^3_x} \frac{2\mu}{|x-y|} |u(y)|^2 \{|u(x)|^2 f(|u(x)|^2) - F(|u(x)|^2)\} dx\, dy\, dt.
\end{aligned}
\tag{4.35}
$$

In particular, we obtain the following space-time $L^4([0,T]\times\mathbb{R}^3)$ estimate,

$$
\int_0^T \int_{\mathbb{R}^3} |u(t,y)|^4 dy\, dt \lesssim \|u_0\|^2_{L^2(\mathbb{R}^3)} \sup_{t\in[0,T]} \|u(t)\|^2_{\dot{H}_x^{1/2}}. \tag{4.36}
$$

4.2. Almost Conservation Law

As we have already mentioned, the solutions of (4.1) with initial data in $H^s(\mathbb{R}^3)$ for $s<1$ can be too irregular to have finite energy. This is the reason for which we can not use the standard arguments based in the conservation law (4.5) to prove global existence and scattering. Our aim will be to control the growth in time of a smoothed version of the solution, but enough close to the first so that we'll be able to control the Sobolev norm of the original solution from energy estimates of the approximation. The principal tool we'll use is the *Littlewood-Paley Theory*, described in Appendix A.3., which allows us to obtain bounds in the spatial side from estimates in the frequence space.

Definition 4.2. *Let $\phi(x,t)$ a solution of* (4.1)-(4.2)*. We define the operator $I_{s,N} = I : H^s(\mathbb{R}^3) \to H^1(\mathbb{R}^3)$, depending on a parameter $N \gg 1$ to be chosen later, by*

$$
\widehat{If}(\xi) := m_N(\xi)\widehat{f}(\xi), \tag{4.37}
$$

where the multiplier $m_N(\xi)$ is smooth, radially symmetric, and nonincreasing in $|\xi|$, and

$$
m_N(\xi) = \begin{cases} 1, & |\xi| \le N, \\ \psi_N(|\xi|), & N \le |\xi| \le 2N, \\ \left(\dfrac{N}{|\xi|}\right)^{1-s}, & |\xi| \ge 2N; \end{cases} \tag{4.38}
$$

with $\psi_N = \psi(\cdot/N)$ as a smooth nexus connecting the two sides of m_N in the proper way.

The following two inequalities relate the energy of $I\phi$ with the size of ϕ and ϕ_0 in $H^s(\mathbb{R}^3)$ and the $L^4(\mathbb{R}^3)$ norm of ϕ.

Proposition 4.3. *Let ϕ be a solution of* (4.1)-(4.2) *and I the operator defined above. Then*

$$
E(I\phi)(t) \lesssim \left(N^{1-s}\|\phi(\cdot,t)\|_{\dot{H}^s(\mathbb{R}^3)}\right)^2 + \|\phi(\cdot,t)\|^4_{L^4(\mathbb{R}^3)}. \tag{4.39}
$$

$$
\|\phi(\cdot,t)\|^2_{H^s(\mathbb{R}^3)} \lesssim E(I\phi)(t) + \|\phi_0\|^2_{L^2(\mathbb{R}^3)}. \tag{4.40}
$$

Proof. By (4.5), we have

$$E(I\phi)(t) = \frac{1}{2}\int_{\mathbb{R}^3} |\nabla_x I\phi(x,t)|^2 dx + \frac{1}{4}\|I\phi(\cdot,t)\|^4_{L^4(\mathbb{R}^3)}.$$

First, by considering separately those frequences $|\xi| \le N$ and $|\xi| \ge N$,

$$\begin{aligned}
\int_{\mathbb{R}^3} |\nabla_x I\phi(x,t)|^2 dx &= \int_{|\xi|\le N} |\xi|^{2-2s}|\xi|^{2s}|\widehat{\phi}(\xi,t)|^2 d\xi + \int_{|\xi|\ge N} |\xi|^2 |m_N(\xi)|^2 |\widehat{\phi}(\xi,t)|^2 d\xi \\
&\le N^{2-2s}\int_{|\xi|\le N} |\xi|^{2s}|\widehat{\phi}(\xi,t)|^2 d\xi + (2N)^{2-2s}\int_{N\le|\xi|\le 2N} |\xi|^{2s}|\widehat{\phi}(\xi,t)|^2 d\xi \\
&\quad + \int_{|\xi|\ge 2N} |\xi|^2 \left(\frac{N}{|\xi|}\right)^{2-2s} |\widehat{\phi}(\xi,t)|^2 d\xi \\
&\lesssim \left(N^{1-s}\|\phi(\cdot,t)\|_{\dot{H}^s(\mathbb{R}^3)}\right)^2.
\end{aligned}$$

Now, we need to see that

$$\|I\phi(\cdot,t)\|^4_{L^4(\mathbb{R}^3)} \lesssim \|\phi(\cdot,t)\|^4_{L^4(\mathbb{R}^3)},$$

which is a consequence of the Hörmander multiplier theorem A.9, once we show that

$$|D^\beta m_N(\xi)| \le C|\xi|^{-\beta}, \text{ for } |\beta| \le 2.$$

We distinguish the following cases:

1. If $|\xi| \le N$, then $D^\beta m_N(\xi) = 0$.
2. If $|\xi| \ge 2N$, then $|D^\beta m_N(\xi)| \sim N^{1-s}|\xi|^{s-1-|\beta|} \le N^{1-s}N^{s-1}|\xi|^{-|\beta|}$.
3. If $N \le |\xi| \le 2N$, then $0 \le |D^\beta m_N(\xi)| = |D^\beta \psi_N(\xi)| = N^{-|\beta|}|D^\beta\psi(\xi)| \lesssim |\xi|^{-|\beta|}$.

So we have proven (4.39).

For the other estimate, note that

$$\begin{aligned}
\|\phi(\cdot,t)\|^2_{H^s(\mathbb{R}^3)} &= \int_{\mathbb{R}^3} (1+|\xi|^2)^s |\widehat{\phi}(\xi,t)|^2 d\xi \\
&\lesssim \|\phi(\cdot,t)\|^2_{L^2(\mathbb{R}^3)} + \int_{|\xi|\le 1} |\xi|^{2s}|\widehat{\phi}(\xi,t)|^2 d\xi + \int_{|\xi|\ge 1} |\xi|^{2s}|\widehat{\phi}(\xi,t)|^2 d\xi \\
&\le 2\|\phi(\cdot,t)\|^2_{L^2(\mathbb{R}^3)} + \int_{1\le|\xi|\le N} |\xi|^2 |\widehat{\phi}(\xi,t)|^2 d\xi \\
&\quad + 2^{2-2s}\int_{N\le|\xi|\le 2N} |\xi|^2|\psi_N(\xi)|^2|\widehat{\phi}(\xi,t)|^2 d\xi + \int_{|\xi|\ge 2N} |\xi|^2\left(\frac{N}{|\xi|}\right)^{2-2s} |\widehat{\phi}(\xi,t)|^2 d\xi \\
&\lesssim E(I\phi)(t) + \|\phi_0\|^2_{L^2(\mathbb{R}^3)}.
\end{aligned}$$

□

The main result at this point is the following.

Proposition 4.4 (Almost Conservation Law)**.** *Assume we have $s > \frac{1}{2}$, $N \gg 1$, $\phi_0 \in \mathcal{C}_0^\infty(\mathbb{R}^3)$, and a solution of* (4.1)-(4.2) *on a time* $[0,T]$ *for which*

$$\|\phi\|_{L^4_{x,t}([0,T]\times\mathbb{R}^3} \lesssim \epsilon. \tag{4.41}$$

Assume in addition that $E(I\phi_0) \lesssim 1$. We conclude that for all $t \in [0,T]$,

$$E(I\phi)(t) = E(I\phi_0) + O(N^{-1+}). \tag{4.42}$$

where $-1+ \equiv -1 + \delta$ for some universal constant $0 < \delta \ll 1$, and the implicit constant in (4.42) *is independent of the length of* $[0,T]$ *and depends only on the constant that bounds* $E(I\phi_0)$.

If one could replace the increment N^{-1+} in $E(I\phi)$ on the right side of (4.42) with $N^{-\alpha}$ for some $\alpha > 0$, one could obtain, by repeating the argument we give below, global well-posedness of (4.1)-(4.2) for all $s > \frac{3+\alpha}{3+2\alpha}$.

In order to prove the previus proposition, we'll need control a local-in-time norm $Z_I(t)$ involving the Schrödinger exponents in (4.3),

$$Z_I(t) \equiv \sup_{q,r \text{ admissible}} \|\nabla I\phi\|_{L^q_t L^r_x([0,t]\times\mathbb{R}^3)}. \tag{4.43}$$

Lemma 4.3. *Consider $\phi(x,t)$ as in* (4.1)-(4.2) *defined on $[0,T] \times \mathbb{R}^3$, with $\phi_0 \in \mathcal{C}_0^\infty(\mathbb{R}^3)$ and*

$$\|\phi\|_{L^4_{x,t}([0,T]\times\mathbb{R}^3)} \leq \epsilon \tag{4.44}$$

for some universal constant ϵ. Assume, too, that $E(I\phi_0) \lesssim 1$. Then for $s > \frac{1}{2}$, and sufficiently large N,

$$Z_I(T) \lesssim 1. \tag{4.45}$$

Proof. Apply ∇I to both sides of (4.1). Choosing $\tilde{q}', \tilde{r}' = \frac{10}{7}$, Strichartz estimates give us that for all $0 \leq t \leq T$,

$$Z_I(t) \lesssim \|\nabla I\phi_0\|_{L^2(\mathbb{R}^3)} + \|\nabla I(|\phi|^2\phi)\|_{L^{10/7}_{x,t}}$$

Now, by fractional Leibniz rule (A.19) for the operator ∇I (the proof can be easily modified for this operator), we obtain

$$Z_I(t) \lesssim \|\nabla I\phi_0\|_{L^2(\mathbb{R}^3)} + \|\nabla I\phi\|_{L^{10/3}_{x,t}([0,t]\times\mathbb{R}^3)} \cdot \|\phi\|^2_{L^5_{x,t}}.$$

Since $q = r = \frac{10}{3}$ are admissible exponents, we have that the $L^{10/3}$ factor is bounded by $Z_I(t)$. We claim that, for N sufficiently large, the remaining $L^5_{x,t}$ factors are bounded by

$$\|\phi\|_{L^5_{x,t}([0,t]\times\mathbb{R}^3)} \lesssim \epsilon^{\delta_1} \cdot (Z_I(t))^{\delta_2}, \tag{4.46}$$

for some $\delta_1, \delta_2 > 0$. Assuming (4.46) for the moment, we conclude, using the hypotesis $E(I\phi_0) \lesssim 1$, that

$$Z_I(t) \lesssim 1 + \epsilon^{2\delta_1}(Z_I(t))^{1+2\delta_2}.$$

For a sufficiently small choice of ϵ (only depending of the exponents δ_1 and δ_2 and the implicit constant), we obtain the bound (4.45) for all $0 \leq t \leq T$, noting that $Z_I(t)$ is a continuous function of t, $0 \le Z_I(t)$ for $0 \le t \le T$, and $f(z) = 1 + \epsilon^{2\delta_1} z^{1+2\delta_2} - Cz$ begins decreasing for $z \ge 0$ and takes negative values for small ϵ.

It remains to prove (4.46). All space-time norms in this proof will be taken on the slab $[0,T] \times \mathbb{R}^3$, even when, for legibility, this isn't explicitly written. Let $\varphi \in \mathcal{C}_0^\infty(\mathbb{R}^3)$ such that

$$\begin{cases} \text{supp } \varphi \subset \{\frac{1}{4} < |\xi| < 2\}, \\ \varphi \equiv 1, \text{ in } \{\frac{1}{2} \le |\xi| \le 1\}. \end{cases}$$

Define $\varphi_k(\xi) = \varphi(\xi/2^k)$ for $k \in \mathbb{Z}$ and

$$\eta_0 = \sum_{k=-\infty}^{k_0} \frac{\varphi_k}{\sum_{i=-\infty}^{\infty} \varphi_i}, \qquad \eta_j = \frac{\varphi_{k_0+j}}{\sum_{i=-\infty}^{\infty} \varphi_i}, \quad j = 1, 2, \ldots$$

where $2^{k_0-1} \le N \equiv N_0 < 2^{k_0}$, and denote $N_j \equiv 2^{k_0+j}$ for $j \ge 1$. Now, we can decompose ϕ in dyadic pieces in the frequency space, writing

$$\phi = \psi_0 + \sum_{j=1}^{\infty} \psi_j$$

with $\widehat{\psi}_j(\xi) = \eta_j(\xi) \cdot \widehat{\phi}(\xi)$, for $j = 0, 1, 2, \ldots$. By construction, we have

$$|\widehat{I\psi_j}(\xi)| \sim \left(\frac{N}{N_j}\right)^{1-s} |\widehat{\psi_j}(\xi)|, \quad j = 0, 1, 2, \ldots$$

so, by the Hörmander multiplier theorem A.9, we obtain that

$$\|I\psi_j\|_{L^{10}_{x,t}} \sim \left(\frac{N}{N_j}\right)^{1-s} \|\psi_j\|_{L^{10}_{x,t}}, \quad j = 0, 1, 2, \ldots$$

and then

$$\|I\psi_j\|_{L^{10}_{x,t}} \sim \begin{cases} \|\psi_j\|_{L^{10}_{x,t}}, & j = 0, \\ N^{1-s}(N_j)^{s-1}\|\psi_j\|_{L^{10}_{x,t}}, & j = 1, 2, \ldots. \end{cases} \tag{4.47}$$

Using Sobolev's inequality in the space-norm, Hörmander theorem once more time, and the admissible exponents $q = 10,\ r = \frac{30}{13}$,

$$\|I\psi_j\|_{L^{10}_{x,t}} \lesssim \|\nabla I\psi_j\|_{L^{10}_t L^{30/13}_x} \lesssim \|\nabla I\phi\|_{L^{10}_t L^{30/13}_x} \lesssim Z_I(T).$$

Rewriting gives

$$\|\psi_j\|_{L^{10}_{x,t}} \lesssim \begin{cases} Z_I(T), & j = 0, \\ N_j^{1-s} N^{s-1} Z_I(T), & j = 1, 2, \ldots. \end{cases} \tag{4.48}$$

Similarly, by Hörmander multiplier theorem,

$$\|\nabla I\psi_j\|_{L^{10/3}_{x,t}} \sim N_j^s N^{1-s} \|\psi_j\|_{L^{10/3}_{x,t}}, \quad j = 1, 2, \ldots.$$

Hence, using the admissible exponents $q = r = \frac{10}{3}$, we get the following $L^{10/3}$ bounds,

$$\|\psi_j\|_{L^{10/3}_{x,t}} \lesssim N^{s-1}(N_j)^{-s} Z_I(T), \quad j \ge 1. \tag{4.49}$$

Now we have the necessarily tools for our desired $L^5_{x,t}$ bound of ϕ. First, apply the triangle inequality to show that

$$\|\phi\|_{L^5_{x,t}} \leq \sum_{j=0}^{\infty} \|\psi_j\|_{L^5_{x,t}}.$$

By Hölder inequality, which allows us to interpolate between the L^{10} and L^4 bounds of (4.48) and (4.44)), and applying Hörmander theorem in order to bound $\|\psi_0\|_{L^p_{x,t}}$ by $\|\phi\|_{L^p_{x,t}}$, we obtain

$$\begin{aligned} \|\psi_0\|_{L^5_{x,t}} &\lesssim \|\psi_0\|^{2/3}_{L^4_{x,t}} \cdot \|\psi_0\|^{1/3}_{L^{10}_{x,t}} \\ &\lesssim \epsilon^{\frac{2}{3}} (Z_I(T))^{\frac{1}{3}} \end{aligned} \tag{4.50}$$

Finally, for $j \geq 1$, interpolation with Hölder inequality between (4.48) and (4.49), yields

$$\begin{aligned} \sum_{j=1}^{\infty} \|\psi_j\|_{L^5_{x,t}} &\lesssim \sum_{j=1}^{\infty} \|\psi_j\|^{1/2}_{L^{10/3}_{x,t}} \cdot \|\psi_j\|^{1/2}_{L^{10}_{x,t}} \\ &\lesssim \sum_{j=1}^{\infty} \left(N^{s-1}(N_j)^{-s} Z_I(T)\right)^{\frac{1}{2}} \cdot \left((N_j)^{1-s} N^{s-1} Z_I(T)\right)^{\frac{1}{2}} \\ &= N^{s-1} Z_I(T) \sum_{j=1}^{\infty} N_j^{\frac{1-2s}{2}}. \end{aligned}$$

Since $s > \frac{1}{2}$, and the N_j are negative powers of 2, the last sum is bounded by a constant only depending on s. Moreover, since $s < 1$, we can choose N sufficiently large, depending on ϵ, to obtain (4.46).

□

Now we can prove the Almost Conservation Law.

Proof of Proposition 4.4. We begin to estimate $E(I\phi)(t)$ in a similar way as we obtained conserved energy for smooth solutions of (4.1). However, $I\phi$ is not a solution, so we have

$$\begin{aligned} \frac{d}{dt} E(I\phi)(t) &= \mathscr{Re} \int_{\mathbb{R}^3} \overline{I\phi_t}(|I\phi|^2 I\phi - \Delta I\phi) dx \\ &= \mathscr{Re} \int_{\mathbb{R}^3} \overline{I\phi_t}(|I\phi|^2 I\phi - \Delta I\phi - iI\phi_t) dx \\ &= \mathscr{Re} \int_{\mathbb{R}^3} \overline{I\phi_t}(|I\phi|^2 I\phi - I(|\phi|^2\phi)) dx \end{aligned}$$

In the next step, we are going to use the following form of Parseval's formula:

$$\int_{\mathbb{R}^d} f_1(x) f_2(x) f_3(x) f_4(x) dx = \int_{\sum_{i=1}^4 \xi_i = 0} \widehat{f_1}(\xi_1)\widehat{f_2}(\xi_2)\widehat{f_3}(\xi_3)\widehat{f_4}(\xi_4) d\sigma.$$

Applying this, and integrating in time, it remains for us to bound

$$\begin{aligned} &E(I\phi)(t) - E(I\phi)(0) = \\ &\mathscr{Re} \int_0^t \int_{\sum_{i=1}^4 \xi_i = 0} \left(1 - \frac{m(\xi_2+\xi_3+\xi_4)}{m(\xi_2)\cdot m(\xi_3)\cdot m(\xi_4)}\right) \widehat{\overline{I\partial_t\phi}}(\xi_1)\widehat{I\phi}(\xi_2)\widehat{\overline{I\phi}}(\xi_3)\widehat{I\phi}(\xi_4). \end{aligned} \tag{4.51}$$

We use equation (4.1) and substitute $\partial_t I(\phi) = i\Delta I\phi - iI(|\phi|^2\phi)$ in (4.51). Our aim is to show that

$$\text{Term}_1 + \text{Term}_2 \lesssim N^{-1+}(Z_I(T))^P, \tag{4.52}$$

for some $P > 0$, where the two terms on the left are

$$\text{Term}_1 \equiv \left| \int_0^T \int_{\sum_{i=1}^4 \xi_i = 0} \left(1 - \frac{m(\xi_2+\xi_3+\xi_4)}{m(\xi_2)\cdot m(\xi_3)\cdot m(\xi_4)}\right) \widehat{(\Delta \overline{I\phi})}(\xi_1)\widehat{I\phi}(\xi_2)\widehat{\overline{I\phi}}(\xi_3)\widehat{I\phi}(\xi_4)\right|,$$

$$\text{Term}_2 \equiv \left| \int_0^T \int_{\sum_{i=1}^4 \xi_i = 0} \left(1 - \frac{m(\xi_2+\xi_3+\xi_4)}{m(\xi_2)\cdot m(\xi_3)\cdot m(\xi_4)}\right) \widehat{(\overline{I(|\phi|^2\phi)})}(\xi_1)\widehat{I\phi}(\xi_2)\widehat{\overline{I\phi}}(\xi_3)\widehat{I\phi}(\xi_4)\right|.$$

Once we prove estimate (4.52), the result follows inmediately by applying Lemma 4.3. In what follows we drop the complex conjugates since they don't affect the analysis used here.

Consider first Term_1. We break ϕ into sum of dyadic constituents ψ_j, each localized with a smooth cutoff function in spatial frequency space to have support $|\xi| \sim N_j \equiv 2^j$, $j = 0, 1, 2, \ldots$.

With some abuse of notation, we will refer to ϕ_i $(= \phi_{i,j})$ as the localization of ϕ in the frequency shell $|\xi_i| \sim N_i$ $(= N_{i,j} \equiv 2^j)$, with $i = 1, 2, 3, 4$ (and $j = 0, 1, 2, \ldots$).

We will conclude that $\text{Term}_1 \leq N^{-1+}$ once we prove

$$\left| \int_0^T \int_{\sum_{i=1}^4 \xi_i = 0} \left(1 - \frac{m(\xi_2+\xi_3+\xi_4)}{m(\xi_2)\cdot m(\xi_3)\cdot m(\xi_4)}\right) \widehat{\Delta I\phi_1}(\xi_1)\widehat{I\phi_2}(\xi_2)\widehat{I\phi_3}(\xi_3)\widehat{I\phi_4}(\xi_4)\right| \lesssim N^{-1+}C(N_1,N_2,N_3,N_4)(Z_I(T))^4 \tag{4.53}$$

where $C(N_1, N_2, N_3, N_4)$ is sufficiently small in each (N_1, N_2, N_3, N_4)-shell to sum over all dyadic shells. By symmetry, we may assume $N_2 \geq N_3 \geq N_4$. We analyze separately the following cases. Here we employ the notation $B \gg A$ denoting $B > K \cdot A$, where K is the implicit constant avoided with the use of $\gtrsim$ in the same context.

Case 1: $N \gg N_2 \geq N_3 \geq N_4$. By the definition of the symbol m in (4.38), we have

$$1 - \frac{m(\xi_2+\xi_3+\xi_4)}{m(\xi_2)\cdot m(\xi_3)\cdot m(\xi_4)} \equiv 0,$$

then the bound (4.53) holds trivially.

Case 2: $N_2 \gtrsim N \gg N_3 \geq N_4$. Since $\sum_i \xi_i = 0$, we have $N_1 \sim N_2$. We aim for (4.53) with

$$C(N_1, N_2, N_3, N_4) = N_2^{0-}. \tag{4.54}$$

With this decay factor, we only have a sum over finite terms in N_3 and N_4, which is of order $\log^2 N$ and can be absorbed within the N^{-1+} term. Moreover, as $N_1 \sim N_2$ and N_i are powers of two, for each N_2 we have a constant number of N_1's. Then, we conclude

$$\sum_{N_1 \sim N_2 \gtrsim N \gg N_3 \geq N_4} N_2^{0-} \lesssim \log^2 N.$$

It remains to show (4.54). By mean value theorem,

$$\left|\frac{m(\xi_2)-m(\xi_2+\xi_3+\xi_4)}{m(\xi_2)}\right| \lesssim \frac{|\nabla m(\xi_2)\cdot(\xi_3+\xi_4)|}{m(\xi_2)} \lesssim \frac{N_3}{N_2} \tag{4.55}$$

Now we use Coifman-Meyer theorem A.10 for multilinear operators, for the symbol

$$\sigma(\xi_2,\xi_3,\xi_4) = \frac{N_2}{N_3}\cdot\frac{m(\xi_2)-m(\xi_2+\xi_3+\xi_4)}{m(\xi_2)}.$$

The necessary bounds (A.15) follow quickly from the definition of m. Then, by applying Hölder's inequality, Theorem A.10 and Hörmander theorem,

$$\begin{aligned}
|\text{left side of (4.53)}| &= \frac{N_3}{N_2}\left|\int_0^T\int_{\sum_i \xi_i=0}\sigma(\xi_1,\xi_2,\xi_3)\widehat{\Delta I\phi_1}(\xi_1)\widehat{I\phi_2}(\xi_2)\widehat{I\phi_3}(\xi_3)\widehat{I\phi_4}(\xi_4)\right|\\
&= \frac{N_3}{N_2}\left|\int_0^T\int_{\mathbb{R}^3}\Lambda[\Delta I\phi_1, I\phi_2, I\phi_3]\cdot I\phi_4\,dx\,dt\right|\\
&\le \frac{N_3}{N_2}\|\Lambda[\Delta I\phi_1, I\phi_2, I\phi_3]\|_{L^{10/9}_{x,t}}\cdot\|I\phi_4\|_{L^{10}_{x,t}}\\
&\lesssim \frac{N_3}{N_2}\|\Delta I\phi_1\|_{L^{10/3}_{x,t}}\cdot\|I\phi_2\|_{L^{10/3}_{x,t}}\cdot\|I\phi_3\|_{L^{10/3}_{x,t}}\cdot\|I\phi_4\|_{L^{10}_{x,t}}\\
&\lesssim \frac{1}{N_2}\|\Delta I\phi_1\|_{L^{10/3}_{x,t}}\cdot\|I\phi_2\|_{L^{10/3}_{x,t}}\cdot\|\nabla I\phi_3\|_{L^{10/3}_{x,t}}\cdot\|I\phi_4\|_{L^{10}_{x,t}}\\
&\lesssim \frac{N_1}{N_2\cdot N_2}(Z_I(T))^4\\
&\lesssim \frac{1}{N_2}(Z_I(T))^4\\
&\le N^{-1+}\cdot N_2^{0-}(Z_I(T))^4,
\end{aligned}$$

where we have used Sobolev inequality and Hörmander theorem for the estimate $\|I\phi_4\|_{L^{10}_{x,t}} \lesssim Z_I(T)$.

Case 3: $N_2 \ge N_3 \gtrsim N$. We use the pointwise estimate

$$\left|1-\frac{m(\xi_2+\xi_3+\xi_4)}{m(\xi_2)m(\xi_3)m(\xi_4)}\right| = \left|\frac{m(\xi_1)(m(\xi_2)m(\xi_3)-1)}{m(\xi_2)m(\xi_3)m(\xi_4)}\right| \lesssim \frac{m(\xi_1)}{m(\xi_2)m(\xi_3)m(\xi_4)} \tag{4.56}$$

The frequency interactions here fall into two subcategories, depending on which frequency is comparable to N_2.

Case 3(a): $N_1 \sim N_2 \ge N_3 \gtrsim N$. We have, by hypothesis, $s > \frac{1}{2}+\delta$ for some small δ. In this case we prove (4.53) with the decay factor

$$|\text{left side of (4.53)}| \lesssim N^{-1+2\delta}N_3^{0-2\delta}(Z_I(T))^4 \tag{4.57}$$

where, in this case, the sum only occurs in $N_3 \gtrsim N$. That is

$$\sum_{\substack{N_3\gtrsim N\\ N_3\ge N_4}} N_3^{0-2\delta} \lesssim \sum_{N_3\lesssim N} N_3^{0-2\delta}\log N_3 \lesssim 1.$$

The sum in $N_1 \sim N_2$ is taking within the integral, as follows. We begin by estimating

$$\frac{1}{m(N_3)m(N_4)}\left|\int_0^T \int_{\sum_i \xi_i=0} \sigma(\xi_1,\xi_2,\xi_3)\widehat{\nabla I\phi}(\xi_1)\widehat{\nabla I\phi}(\xi_2)\widehat{I\phi_3}(\xi_3)\widehat{I\phi_4}(\xi_4)\right|, \tag{4.58}$$

where the symbol σ is now defined by

$$\sigma(\xi_1,\xi_2,\xi_3) = m(N_3)m(N_4)\cdot\left(1-\frac{m(\xi_2+\xi_3+\xi_4)}{m(\xi_2)m(\xi_3)m(\xi_4)}\right)\cdot\frac{|\xi_1|}{|\xi_2|}\cdot\psi\left(\frac{|\xi_1|}{|\xi_2|}\right)\cdot\eta\left(\frac{|\xi_2|}{|\xi_3|}\right),$$

with ψ a smooth cut-off function in the frequency shell $|\xi|\sim 1$ and η another cut-off in the shell $|\xi|\gtrsim 1$. Note that this symbol satisfy the hypothesis (A.15) in the considered region, since using (4.56) we have

$$\begin{aligned}|\sigma(\xi_1,\xi_2,\xi_3)| &\lesssim \frac{m(\xi_1)}{m(\xi_2)}\cdot\frac{|\xi_1|}{|\xi_2|}\cdot\psi\left(\frac{|\xi_1|}{|\xi_2|}\right)\cdot\eta\left(\frac{|\xi_2|}{|\xi_3|}\right)\\ &\lesssim 1.\end{aligned}$$

The important fact is that we have replaced the projections ϕ_1 and ϕ_2 by the cut-off functions ψ and η so, in this case, we don't have to sum in N_1 and N_2. Note also that we have left two terms of the form $\widehat{\nabla I\phi}$ in the integrand, multiplying the symbol by $\frac{|\xi_1|}{|\xi_2|}$. Applying Hölder's inequality, Theorem A.10, Hörmander theorem and Sobolev embedding once again, we obtain

$$\begin{aligned}(4.58) &\leq \frac{1}{m(N_3)m(N_4)}\|\Lambda(\nabla I\phi,\nabla I\phi, I\phi_3)\|_{L^{10/9}_{x,t}}\cdot\|I\phi_4\|_{L^{10}_{x,t}}\\ &\lesssim \frac{1}{m(N_3)m(N_4)}\|\nabla I\phi\|_{L^{10/3}_{x,t}}\cdot\|\nabla I\phi\|_{L^{10/3}_{x,t}}\cdot\|I\phi_3\|_{L^{10/3}_{x,t}}\cdot\|I\phi_4\|_{L^{10}_{x,t}}\\ &\lesssim \frac{1}{m(N_3)m(N_4)N_3}(Z_I(T))^4.\end{aligned}$$

It remains to show

$$\frac{N^{1-2\delta}N_3^{2\delta}}{m(N_3)m(N_4)N_3}\lesssim 1 \tag{4.59}$$

In the proof of this estimate and the analogous in the case 3(b) we use that: for any $p\geq \frac{1}{2}-\delta$, the function $m(x)|x|^p$ is increasing. Then

$$\begin{aligned}\text{left side of (4.59)} &\lesssim \frac{N^{1-2\delta}N_3^{2\delta}}{(m(N_3))^2N_3}\\ &= \frac{N^{1-2\delta}N_3^{2\delta}}{m(N_3)N_3^{\frac{1}{2}-\delta}m(N_3)N_3^{\frac{1}{2}-\delta}N_3^{2\delta}}\\ &\lesssim \frac{N^{1-2\delta}N_3^{2\delta}}{N^{1-2\delta}N_3^{2\delta}}.\end{aligned}$$

Case 3(b): $N_2\sim N_3\gtrsim N$. We aim in this case for the decay factor

$$|\text{left side of (4.53)}|\lesssim N^{-1+2\delta}N_2^{0-2\delta}(Z_I(T))^4 \tag{4.60}$$

where δ is as in case 3(a) above. Then we can sum in all the N_i, using that for each N_2 there is a constant number of N_3's (since $N_3 \sim N_2$ and the N_i are powers of 2) and the fact $N_3 \geq N_4 \sim N_1$:

$$\sum_{\substack{N_2 \sim N_3 \gtrsim N \\ N_3 \geq N_4 \sim N_1}} N_2^{0-2\delta} \lesssim \sum_{\substack{N_3 \gtrsim N \\ N_3 \geq N_4}} N_3^{0-} \lesssim \sum_{N_3 \gtrsim N} N_3^{0-} \log N_3 \lesssim 1.$$

Once again, for obtaining this decay, we repeat the same argument, applying Hölder's inequality, Theorem A.10 and Hörmander theorem with the symbol

$$\sigma(\xi_1, \xi_2, \xi_3) = \frac{m(N_2)m(N_3)m(N_4)}{m(N_1)} \cdot \left(1 - \frac{m(\xi_2 + \xi_3 + \xi_4)}{m(\xi_2)m(\xi_3)m(\xi_4)}\right).$$

To conclude, we just have to show that

$$\begin{aligned}
\frac{m(N_1)N_1 N^{1-2\delta} N_2^{2\delta}}{m(N_2)m(N_3)m(N_4)N_2 N_3} &\lesssim \frac{m(N_1)N_1 N^{1-2\delta} N_2^{2\delta}}{(m(N_2))^3 N_2 N_2} \\
&\lesssim \frac{m(N_2)N_2 N^{1-2\delta} N_2^{2\delta}}{(m(N_2))^3 N_2 N_2} \\
&= \frac{N^{1-2\delta} N_2^{2\delta}}{(m(N_2))^2 N_2} \\
&\leq \frac{N^{1-2\delta} N_2^{2\delta}}{N_2^{2\delta} N^{1-2\delta}} \\
&\leq 1.
\end{aligned}$$

We conclude the proof of the proposition by bounding Term_2. When decomposing the integrand of Term_2 in the frequency space, write N_1 for the dyadic frequency into which we project the nonlinear factor $I(\phi^3)$. The analysis above for Term_1 applies unmodified to Term_2 once we prove the following:

Lemma 4.4. *Assume ϕ, T, $Z_I(T)$ and N_1 are as defined above, and P_{N_1} the Littlewood-Paley projection onto the N_1 frequency shell. Then*

$$\|P_{N_1}(I(\phi^3))\|_{L^{10/3}_{x,t}([0,T]\times\mathbb{R}^3)} \lesssim N_1 (Z_I(T))^3. \tag{4.61}$$

Proof. We write $\phi = \phi_L + \phi_H$, where

$$\begin{aligned}
\text{supp } \widehat{\phi}_L(\xi, t) &\subseteq \{|\xi|^2 < 2\}, \\
\text{supp } \widehat{\phi}_H(\xi, t) &\subseteq \{|\xi|^2 > 1\}.
\end{aligned}$$

Consider first the case when all the three factors in (4.61) are ϕ_L. By Hörmander theorem and Sobolev embedding,

$$\begin{aligned}
\|P_{N_1}(I(\phi_L^3))\|_{L^{10/3}_{x,t}} &\lesssim \|\phi_L\|^3_{L^{10}_{x,t}} \\
&= \|I\phi_L\|^3_{L^{10}_{x,t}} \\
&\leq \|\nabla I \phi_L\|^3_{L^{10}_t L^{30/13}_x} \\
&\leq (Z_I(T))^3 \leq N_1 (Z_I(T))^3,
\end{aligned}$$

since $N_1 \geq 1$.

Now, we consider the case when all the three factors in (4.61) are ϕ_H. By Hörmander theorem, Sobolev embedding and the fractional Leibniz rule (A.19),

$$\begin{aligned}
\left\|\frac{1}{N_1} P_{N_1} I(\phi_H^3)\right\|_{L^{10/3}_{x,t}} &\lesssim \|\nabla^{-1} P_{N_1} I(\phi_H^3)\|_{L^{10/3}_{x,t}} \\
&\lesssim \|\nabla^{1/2} I(\phi_H^3)\|_{L_t^{10/3} L_x^{10/8}} \\
&\lesssim \|\nabla^{1/2} I\phi_H\|_{L_t^{10} L_x^{30/8}} \cdot \|\phi_H^2\|_{L_t^{10/2} L_x^{30/16}} \\
&= \|\nabla^{1/2} I\phi_H\|_{L_t^{10} L_x^{30/8}} \cdot \|\phi_H\|^2_{L_t^{10/2} L_x^{30/8}} \\
&\lesssim \|\nabla^{1/2} I\phi_H\|^3_{L_t^{10} L_x^{30/8}} \\
&\lesssim \|\nabla I\phi_H\|^3_{L_t^{10} L_x^{30/13}} \\
&\lesssim (Z_I(T))^3.
\end{aligned}$$

The remaining terms are bounded using similar arguments, including fractional Leibniz rule (A.19):

$$\begin{aligned}
\left\|\frac{1}{N_1} P_{N_1} I(\phi_H \cdot \phi_H \cdot \phi_L)\right\|_{L^{10/3}_{x,t}} &\lesssim \|\nabla^{1/2} I(\phi_H \cdot \phi_H \cdot \phi_L)\|_{L_t^{10/3} L_x^{10/8}} \\
&\lesssim \|\nabla^{1/2} I(\phi_H^2)\|_{L_t^5 L_x^{10/7}} \cdot \|\phi_L\|_{L_t^{10} L_x^{10}} \\
&\quad + \|\phi_H^2\|_{L_t^5 L_x^{30/16}} \cdot \|\nabla^{1/2} I\phi_L\|_{L_t^{10} L_x^{30/8}} \\
&\lesssim \|\nabla^{1/2} I\phi_H\|_{L_t^{10} L_x^{30/8}} \cdot \|\phi_H\|_{L_t^{10} L_x^{30/13}} \cdot \|\phi_L\|_{L_t^{10} L_x^{10}} \\
&\quad + \|\phi_H\|_{L_t^{10} L_x^{30/8}} \cdot \|\phi_H\|_{L_t^{10} L_x^{30/8}} \cdot \|\nabla^{1/2} I\phi_L\|_{L_t^{10} L_x^{30/8}} \\
&\lesssim \|\nabla I\phi_H\|_{L_t^{10} L_x^{30/13}} \cdot \|\nabla I\phi_H\|_{L_t^{10} L_x^{30/13}} \cdot \|I\phi_L\|_{L_t^{10} L_x^{10}} \\
&\quad + \|\nabla^{1/2} I\phi_H\|_{L_t^{10} L_x^{30/8}} \cdot \|\nabla^{1/2} I\phi_H\|_{L_t^{10} L_x^{30/8}} \cdot \|\nabla I\phi_L\|_{L_t^{10} L_x^{30/13}} \\
&\lesssim (Z_I(T))^3.
\end{aligned}$$

Finally,

$$\begin{aligned}
\left\|\frac{1}{N_1} P_{N_1} I(\phi_H \cdot \phi_L \cdot \phi_L)\right\|_{L^{10/3}_{x,t}} &\lesssim \|\phi_H \cdot \phi_L \cdot \phi_L\|_{L_t^{10/3} L_x^{30/19}} \\
&\lesssim \|\phi_H\|_{L_t^{10} L_x^{30/13}} \cdot \|\phi_L\|_{L_t^{10} L_x^{10}} \cdot \|\phi_L\|_{L_t^{10} L_x^{10}} \\
&\lesssim \|\nabla I\phi_H\|_{L_t^{10} L_x^{30/13}} \cdot \|\nabla I\phi_L\|^2_{L_t^{10} L_x^{30/13}} \\
&\lesssim (Z_I(T))^3.
\end{aligned}$$

$\square$

4.3. Global Well-posedness Theorem

We now use interaction Morawetz estimate (4.36) and Almost Conservation Law 4.4 proved above. Combining this results with a scaling argument, we prove the following statement giving uniform bounds for smooth solutions of (4.1)-(4.2) in terms of the rough norm of the initial data:

Proposition 4.5. *Suppose $\phi(x,t)$ is a global-in-time solution to* (4.1)-(4.2) *from data $\phi_0 \in C_0^\infty(\mathbb{R}^3)$. Then so long as $s > \frac{4}{5}$, we have*

$$\|\phi\|_{L^4([0,\infty)\times\mathbb{R}^3)} \leq C(\|\phi_0\|_{H^s(\mathbb{R}^3)}), \tag{4.62}$$

$$\sup_{0\leq t<\infty} \|\phi(t)\|_{H^s(\mathbb{R}^3)} \leq C(\|\phi_0\|_{H^s(\mathbb{R}^3)}). \tag{4.63}$$

As mentionated at Section 3.1, the energy conservation for H^s solutions with $s \geq 1$ and the local-in-time well-posedness of (4.1)-(4.2) from data in $H^s(\mathbb{R}^3)$, $s > \frac{1}{2}$, imply that the solution ϕ considered here is smooth and exists globally in time. We will use a density argument in $H^s(\mathbb{R}^3)$, combined with the local existence theorem, to prove the global well-posedness portion of Theorem 4.1.

By the invariant properties of the equation (4.1), it's easy to show that if ϕ is a solution of (4.1), then so is

$$\phi^{(\lambda)}(x,t) \equiv \frac{1}{\lambda}\phi\left(\frac{x}{\lambda}, \frac{t}{\lambda^2}\right). \tag{4.64}$$

The following lemma allows us to make $E(I\phi_0^{(\lambda)})$ small with a proper choice of λ.

Lemma 4.5. *Let $\frac{1}{2} < s < 1$ and $\phi_0 \in H^s(\mathbb{R}^3)$, then*

$$E(I\phi_0^{(\lambda)}) \equiv \frac{1}{2}\|\nabla I\phi_0^{(\lambda)}\|^2_{L^2(\mathbb{R}^3)} + \frac{1}{4}\|I\phi_0^{(\lambda)}\|^4_{L^4(\mathbb{R}^3)} \leq \frac{1}{2}$$

for $\lambda \approx N^{\frac{1-s}{s-1/2}}$, depending on $\|\phi_0\|_{H^s(\mathbb{R}^3)}$.

Proof. First, we show that

$$\|\nabla I\phi_0^{(\lambda)}\|^2_{L^2(\mathbb{R}^3)} \lesssim \left(N^{1-s}\lambda^{1/2-s}\|\phi_0\|_{H^s(\mathbb{R}^3)}\right)^2.$$

We have

$$\begin{aligned}
\|\nabla I\phi_0^{(\lambda)}\|^2_{L^2(\mathbb{R}^3)} &= \int_{\mathbb{R}^3} |\xi|^2 m_N^2(\xi)|\widehat{\phi_0^{(\lambda)}}|^2 d\xi \\
&= \lambda \int_{\mathbb{R}^3} \left|\frac{\eta}{\lambda}\right|^2 m_N^2\left(\frac{\eta}{\lambda}\right)|\widehat{\phi_0}(\eta)|^2\, d\eta.
\end{aligned}$$

By considering separately $|\frac{\eta}{\lambda}| \leq N$ and $|\frac{\eta}{\lambda}| \geq N$, we obtain

$$\begin{aligned}
\lambda \int_{|\frac{\eta}{\lambda}|\leq N} \left|\frac{\eta}{\lambda}\right|^2 m_N^2\left(\frac{\eta}{\lambda}\right)|\widehat{\phi_0}(\eta)|^2\, d\eta &\leq \lambda N^{2-2s} \int_{|\frac{\eta}{\lambda}|\leq N} \left(\frac{1}{\lambda}\right)^{2s} |\eta|^{2s}|\widehat{\phi_0}(\eta)|^2\, d\eta \\
&\leq \lambda^{1-2s} N^{2-2s} \int_{|\frac{\eta}{\lambda}|\leq N} |\eta|^{2s}|\widehat{\phi_0}(\eta)|^2\, d\eta
\end{aligned}$$

and

$$\lambda \int_{|\frac{\eta}{\lambda}|\geq N} \left|\frac{\eta}{\lambda}\right|^2 m_N^2\left(\frac{\eta}{\lambda}\right) |\widehat{\phi_0}(\eta)|^2\, d\eta \leq (2N)^{2-2s}\lambda^{1-2s}\int_{N\leq|\frac{\eta}{\lambda}|\leq 2N} |\eta|^{2s}|\widehat{\phi_0}(\eta)|^2\, d\eta$$

$$+ N^{2-2s}\lambda^{1-2s}\int_{2N\leq|\frac{\eta}{\lambda}|} |\eta|^{2s}|\widehat{\phi_0}(\eta)|^2\, d\eta,$$

respectively, so taking $\lambda \approx N^{\frac{1-s}{s-1/2}}$ yields the desired bound.

It remains to estimate the term $\|I\phi_0^{(\lambda)}\|^4_{L^4(\mathbb{R}^3)}$. Here, by using Plancherel and breaking the integral in the frequencies $|\xi| \leq \frac{1}{\lambda}$, $\frac{1}{\lambda} \leq |\xi| \leq N$ and $|\xi| \geq N$ (denoting these projections P_1, P_2, and P_3 respectively),

$$\begin{aligned}\|IP_1\phi_0^{(\lambda)}\|^4_{L^4_x} &\lesssim \|\nabla^{3/4}IP_1\phi_0^{(\lambda)}\|^4_{L^2_x}\\ &= \lambda^{-1}\left(\int_{|\eta|\leq 1}|\eta|^{3/2}|\widehat{\phi_0}(\eta)|^2\, d\eta\right)^2\\ &\leq \lambda^{-1}\|\phi_0\|^4_{H^s(\mathbb{R}^3)}.\end{aligned}$$

The P_2 case is completely analogous for the case $s > \frac{4}{5} > \frac{3}{4}$, but indeed in the case $\frac{1}{2} < s \leq \frac{3}{4}$ we obtain the same estimate by taking our $\lambda \approx N^{\frac{1-s}{s-1/2}}$. In fact,

$$\begin{aligned}\|IP_2\phi_0^{(\lambda)}\|^4_{L^4_x} &\lesssim \|\nabla^{3/4}IP_2\phi_0^{(\lambda)}\|^4_{L^2_x}\\ &= \lambda^{-1}\left(\int_{1\leq|\eta|\leq\lambda N}|\eta|^{3/2-2s}|\eta|^{2s}|\widehat{\phi_0}(\eta)|^2\, d\eta\right)^2\\ &\leq (\lambda^{-1/2}N^{3/2-2s}\lambda^{3/2-2s})^2\left(\int_{1\leq|\eta|\leq\lambda N}|\eta|^{2s}|\widehat{\phi_0}(\eta)|^2\, d\eta\right)^2\\ &\leq (\lambda^{1-2s}N^{2-2s})^2\|\phi_0\|^4_{H^s(\mathbb{R}^3)}.\end{aligned}$$

Finally,

$$\begin{aligned}\|IP_3\phi_0^{(\lambda)}\|^4_{L^4_x} &\lesssim \|\nabla^{3/4}IP_3\phi_0^{(\lambda)}\|^4_{L^2_x}\\ &\lesssim (\lambda^{1-2s}N^{2-2s})^2\left(\int_{|\eta|\geq\lambda N}|\eta|^{2s}|\widehat{\phi_0}(\eta)|^2\, d\eta\right)^2\end{aligned}$$

and we conclude once more time by taking $\lambda \approx N^{\frac{1-s}{s-1/2}}$. □

Proof of Proposition 4.5. Choosing λ as in the previus lemma, for some constant $C_1 = C(\|\phi_0\|_{H^s(\mathbb{R}^3)})$ to be chosen shortly, define

$$W \equiv \{T : \|\phi^{(\lambda)}\|_{L^4([0,T]\times\mathbb{R}^3)} \leq C_1\lambda^{3/8}\}. \tag{4.65}$$

We claim that the set W of times for which (4.62) holds is all of $[0,\infty]$. The set W is clearly nonempty and closed, since $\|\phi^{(\lambda)}\|_{L^4([0,T]\times\mathbb{R}^3)}$ is contituous in time. It sufficies then to show it is open. By continuous dependence on time of $\|\phi^{(\lambda)}\|_{L^4([0,T]\times\mathbb{R}^3)}$, if $T_1 \in W$, then for some $T_2 > T_1$ sufficiently close to T_1 we have

$$\|\phi^{(\lambda)}\|_{L^4_{x,t}([0,T_2]\times\mathbb{R}^3)} \leq 2C_1\lambda^{3/8}. \tag{4.66}$$

We claim $T_0 \in W$. By (4.36),

$$\|\phi^{(\lambda)}\|_{L^4_{x,t}([0,T_2]\times\mathbb{R}^3)} \lesssim \|\phi_0^{(\lambda)}\|_{L^2_x}^{1/2} \cdot \sup_{0\leq t\leq T_2} \|\phi^{(\lambda)}(t)\|_{\dot{H}^{1/2}(\mathbb{R}^3)}^{1/2} \tag{4.67}$$

$$\leq C(\|\phi_0\|_{L^2_x})\lambda^{\frac{1}{4}} \cdot \sup_{0\leq t\leq T_2} \|\phi^{(\lambda)}(t)\|_{\dot{H}^{1/2}(\mathbb{R}^3)}^{1/2}. \tag{4.68}$$

To bound the second factor in (4.68), decompose $\phi^{(\lambda)}(t)$ as

$$\phi^{(\lambda)}(t) = P_{\leq N}\phi^{(\lambda)}(t) + P_{\geq N}\phi^{(\lambda)}(t). \tag{4.69}$$

That is, a sum of functions supported on frequencies $|\xi| \leq N$ and $|\xi| \geq N$, respectively. Now, by Plancherel, Cauchy Schwartz (that allows us interpolating between L^2 and $\dot{H}^1$), L^2 conservation law and the fact that I is the identity on the low frequencies give us

$$\begin{aligned}
\|P_{\leq N}\phi^{(\lambda)}(t)\|_{\dot{H}^{1/2}} &\lesssim \|P_{\leq N}\phi^{(\lambda)}(t)\|_{L^2_x}^{1/2} \cdot \|P_{\leq N}\phi^{(\lambda)}(t)\|_{\dot{H}^1_x}^{1/2} \\
&\lesssim \|\phi_0^{(\lambda)}\|_{L^2_x}^{1/2} \cdot \|IP_{\leq N}\phi^{(\lambda)}(t)\|_{\dot{H}^1_x}^{1/2} \\
&\leq C(\|\phi_0\|_{L^2_x})\lambda^{\frac{1}{4}}\|I\phi^{(\lambda)}(t)\|_{\dot{H}^1_x}^{1/2}.
\end{aligned} \tag{4.70}$$

We interpolate the high-frequency constituent between $\dot{H}^s_x$ and L^2_x (using Hölder inequality), and use the definition of I to get

$$\begin{aligned}
\|P_{\geq N}\phi^{(\lambda)}(t)\|_{\dot{H}^{1/2}} &\lesssim \|P_{\geq N}\phi^{(\lambda)}(t)\|_{L^2_x}^{1-1/(2s)} \cdot \|P_{\geq N}\phi^{(\lambda)}(t)\|_{\dot{H}^1_x}^{1/(2s)} \\
&= \|P_{\geq N}\phi^{(\lambda)}(t)\|_{L^2_x}^{1-1/(2s)} \cdot N^{\frac{s-1}{2s}}\|IP_{\geq N}\phi^{(\lambda)}(t)\|_{\dot{H}^1_x}^{1/(2s)} \\
&\leq C(\|\phi_0\|_{L^2_x})\lambda^{\frac{1}{2}\frac{2s-1}{2s}}N^{\frac{s-1}{2s}}\|I\phi^{(\lambda)}(t)\|_{\dot{H}^1_x}^{1/2} \\
&= C(\|\phi_0\|_{L^2_x})\|I\phi^{(\lambda)}(t)\|_{\dot{H}^1_x}^{1/2},
\end{aligned} \tag{4.71}$$

where we've used both L^2 conservation law and our choice $\lambda \approx N^{\frac{1-s}{s-1/2}}$, which implies that

$$\lambda^{\frac{1}{2}\frac{2s-1}{2s}}N^{\frac{s-1}{2s}} \approx 1.$$

Putting together (4.71), (4.70), (4.69) and (4.68) gives

$$\begin{aligned}
&\|\phi^{(\lambda)}\|_{L^4_{x,t}([0,T_2]\times\mathbb{R}^3)} \\
&\quad\leq C(\|\phi_0\|_{L^2_x})\left(\lambda^{\frac{3}{8}}\sup_{0\leq t\leq T_2}\|I\phi^{(\lambda)}(t)\|_{\dot{H}^1_x}^{1/4} + \sup_{0\leq t\leq T_2}\|I\phi^{(\lambda)}(t)\|_{\dot{H}^1_x}^{1/(4s)}\right),
\end{aligned} \tag{4.72}$$

and note the fact that $C(\|\phi_0\|_{L^2_x})$ doesn't depend on T_1 or T_2.

We conclude $T_2 \in W$ if we establish

$$\sup_{0 \leq t \leq T_2} \|I\phi^{(\lambda)}(t)\|_{\dot{H}^1_x} \leq 1 \tag{4.73}$$

since we then take C_1 larger than twice the constant $C(\|\phi_0\|_{L^2_x})$ appearing in (4.72).

Since $\|\phi^{(\lambda)}\|_{L^4_{x,t}([0,T_2]\times\mathbb{R}^3)} \leq 2C_1\lambda^{\frac{3}{8}}$, we can divide the interval $[0, T_2]$ into subintervals $I_j = [t_{j-1}, t_j]$, for $j = 1, \dots, L$ such that

$$\|\phi^{(\lambda)}\|_{L^4_{x,t}(I_j\times\mathbb{R}^3)} \leq \epsilon, \quad j = 1, \dots L.$$

Then, applying Almost Conservation Law, we have

$$\sup_{0\leq t\leq t_1} \|\nabla I\phi^{(\lambda)}(t)\|_{L^2(\mathbb{R}^3)} \leq E(I\phi_0^{(\lambda)}) + CN^{-1+}$$

$$E(I\phi^{(\lambda)})(t_1) \leq E(I\phi_0^{(\lambda)}) + CN^{-1+},$$

where C doesn't depend on N. If we get $E(I\phi_0^{(\lambda)}) + CN^{-1+} \leq 1$, we can repeat the argument to obtain

$$\begin{aligned}\sup_{t_1\leq t\leq t_2} \|\nabla I\phi^{(\lambda)}(t)\|_{L^2(\mathbb{R}^3)} &\leq E(I\phi^{(\lambda)})(t_1) + CN^{-1+}\\ &\leq E(I\phi_0^{(\lambda)}) + 2CN^{-1+}\\ E(I\phi^{(\lambda)})(t_2) &\leq E(I\phi_0^{(\lambda)}) + 2CN^{-1+},\end{aligned}$$

It's clear now that if we can show $CLN^{-1+} \leq \frac{1}{2}$, we may repeat the argument above L times to get

$$\sup_{0\leq t\leq T_2} \|\nabla I\phi^{(\lambda)}(t)\|_{L^2(\mathbb{R}^3)} \leq E(I\phi_0^{(\lambda)}) + CLN^{-1+}$$

and then (4.73) follows. But, since $L\epsilon \leq (2C_1)^4\lambda^{3/2}$, taking $L \approx \lambda^{\frac{3}{2}}$ and $\lambda \approx N^{\frac{1-s}{s-1/2}}$, we need

$$\left(N^{\frac{1-s}{s-1/2}}\right)^{\frac{3}{2}} \cdot N^{-1+} \ll \frac{1}{2},$$

and this is possible since for $s > \frac{4}{5}$ the exponent on the left is negative. Notice that (4.63) holds on the set W using (4.73), the definition of I, and L^2 conservation. $\square$

We conclude the proff of the global well-posedness theorem by showing the density argument already mentioned. Let $\phi_0 \in H^s(\mathbb{R}^3)$ and take $\phi_0^k \in \mathcal{C}_0^\infty$ such that

$$\|\phi_0^k - \phi_0\|_{H^s(\mathbb{R}^3)} \to 0, \quad \text{as } k \to \infty.$$

By the local existence theorem, there exists a time T in which the solution $\phi(x,t)$ of (4.1)-(4.2) corresponding to ϕ_0 belongs to $H^s(\mathbb{R}^3)$. Moreover, by the continuous dependence of initial data, we have

$$\sup_{0\leq t\leq T'} \|\phi^k(t) - \phi(t)\|_{H^s(\mathbb{R}^3)} \leq C_{T'}\|\phi_0^k - \phi_0\|_{H^s(\mathbb{R}^3)} \to 0,$$

for all $T' < T$. Then

$$\begin{aligned}\sup_{0\le t\le T'} \|\phi(t)\|_{H^s(\mathbb{R}^3)} &\le \sup_{0\le t\le T'} \|\phi^k(t) - \phi(t)\|_{H^s(\mathbb{R}^3)} + \sup_{0\le t\le T'} \|\phi^k(t)\|_{H^s} \\ &\le C_{T'} \|\phi_0^k - \phi_0\|_{H^s(\mathbb{R}^3)} + C_k(\|\phi_0^k\|_{H^s(\mathbb{R}^3)}) \\ &\le C(\|\phi_0\|_{H^s(\mathbb{R}^3)}),\end{aligned}$$

for sufficiently large k and all $T' < T$, since the constant $C(\|\phi_0\|_{H^s(\mathbb{R}^3)})$ in (4.63) depends continuously on the size of ϕ_0 in $H^s(\mathbb{R}^3)$. Then we can extend the solution ϕ over all $t \in [0,\infty)$, and (4.63) holds on all $H^s(\mathbb{R}^3)$.

4.4. Scattering for the Solutions

Asymptotic completeness will follow quickly once we establish a uniform bound of the form

$$Z(t) \equiv \sup_{q,r \text{ admissible}} \|\langle\nabla\rangle^s \phi\|_{L_t^q L_x^r([0,t]\times\mathbb{R}^3)} \tag{4.74}$$

$$\le C(\|\phi_0\|_{H^s(\mathbb{R}^3)}). \tag{4.75}$$

By (4.62), we decompose the time interval $[0,\infty)$ into a finite number of disjoint intervals $J_1, \dots, J_K$ where for $i = 1, \dots, K$ we have

$$\|\phi\|_{L^4_{x,t}(J_i\times\mathbb{R}^3)} \le \epsilon \tag{4.76}$$

for $\epsilon = \epsilon(\|\phi\|_{H^s(\mathbb{R}^3)})$ to be choosen early. Apply $\langle\nabla\rangle^s$ to both sides of (4.1). Choosing $\widetilde{q}', \widetilde{r}' = \frac{10}{7}$, the Strichartz inequalities (2.5) and (2.9) gives us that for all $t \in J_1$,

$$Z(t) \lesssim \|\langle\nabla\rangle^s \phi_0\|_{L^2(\mathbb{R}^3)} + \|\langle\nabla\rangle^s(\phi\phi\phi)\|_{L^{10/7}_{t,x}([0,t]\times\mathbb{R}^3)}.$$

Using the fractional Leibniz rule (A.19) we obtain

$$\|\langle\nabla\rangle^s(\phi\phi\phi)\|_{L^{10/7}_{t,x}([0,t]\times\mathbb{R}^3)} \lesssim \|\langle\nabla\rangle^s\phi\|_{L^{10/3}_{t,x}([0,t]\times\mathbb{R}^3)} \|\phi\|^2_{L^5_{x,t}}.$$

The factor ending up in $L^{10/3}$ is bounded by $Z(t)$. The remaining $L^5_{t,x}$ factors are bounded by interpolating with the Hölder inequality between $\|\phi\|_{L^4_{t,x}}$ and $\|\phi\|_{L^6_{t,x}}$. The latter norm is bounded by $Z(t)$ using Sobolev embedding:

$$\|\phi\|_{L^6_{t,x}} \lesssim \|\langle\nabla\rangle^{2/3}\phi\|_{L^6_t L^{18/7}_x} \le Z(t),$$

noting that $s > \frac{4}{5} > \frac{2}{3}$. We conclude that

$$Z(t) \lesssim \|\phi_0\|_{H^s(\mathbb{R}^3)} + \epsilon^{4/5} Z(t)^{1+6/5}.$$

For a sufficiently small choice of ϵ, this bound yields (4.75) for all $t \in J_1$. Since we are assuming the bound (4.63), we may repeat this argument to control the remaining intervals J_i.

From the inequality (4.75), we prove asymptotic completeness. Given $\phi_0 \in H^s(\mathbb{R}^3)$, we look for a ϕ^+ satisfying (4.6). Set

$$\phi^+ \equiv \phi_0 - i\int_0^\infty S^L(-\tau)(|\phi|^2\phi)d\tau. \tag{4.77}$$

To make sense to this expression, we have to show that the integral on the right-hand side converges in $H^s(\mathbb{R}^3)$. Equivalently, we want

$$\lim_{t\to\infty}\left\|\int_t^\infty \langle\nabla\rangle^s S^L(-\tau)(|\phi|^2\phi)d\tau\right\|_{L^2(\mathbb{R}^3)}=0. \tag{4.78}$$

To prove (4.78), test the time integral against an arbitrary $L^2(\mathbb{R}^3)$ function $F(x)$, with $\|F(x)\|_{L^2(\mathbb{R}^3)}\le 1$. Using the fractional Leibniz rule,

$$\begin{aligned}
&\sup_{\|F(x)\|_{L^2(\mathbb{R}^3)}\le 1}\left\langle F(x),\int_t^\infty \langle\nabla\rangle^s S^L(-\tau)(|\phi|^2\phi)d\tau\right\rangle\\
&\qquad\approx \sup_{\|F(x)\|_{L^2(\mathbb{R}^3)}\le 1}\langle S^L(\tau)F(x),(\nabla^s\phi)\phi\phi\rangle_{L^2_{x,t}([t,\infty)\times\mathbb{R}^3)}\\
&\qquad\le \sup_{\|F(x)\|_{L^2(\mathbb{R}^3)}\le 1}\|S^L(\tau)F(x)\|_{L^{10/3}_{x,t}}\|\nabla^s\phi\|_{L^{10/3}_{x,t}}\|\phi\|^2_{L^5_{x,t}([t,\infty)\times\mathbb{R}^3)}\\
&\qquad\to 0,
\end{aligned}$$

where in the last step we have used (4.75) and the argument with the interpolation and ϵ to bound the $L^5_{x,t}$ term. Note that the convergence is uniform in F since, by Strichartz inequality (2.5),

$$\|S^L(\tau)F(x)\|_{L^{10/3}_{x,t}}\lesssim\|F(x)\|_{L^2(\mathbb{R}^3)}\le 1.$$

This yields (4.78). With this,

$$\begin{aligned}
\lim_{t\to\infty}\|S^L(t)\phi^+-\phi(t)\|_{H^s(\mathbb{R}^3)}&=\lim_{t\to\infty}\left\|\langle\nabla\rangle^s S^L(t)\int_t^\infty S^L(-\tau)(|\phi|^2\phi)d\tau\right\|_{L^2(\mathbb{R}^3)}\\
&=0.
\end{aligned}$$

For completeness, we prove the existence of wave operators on $H^s(\mathbb{R}^3)$. Given $\phi^+\in H^s(\mathbb{R}^3)$, we are looking for a solution $\phi(x,t)$ of (4.1) and data ϕ_0 that, heuristically at least, satisfy

$$\begin{aligned}
\phi(x,t)&=S^L(t)\phi_0-i\int_0^t S^L(t-\tau)|\phi|^2\phi d\tau\\
&=S^L(t)(S^{NL}(-\infty)S^L(\infty)\phi^+)-i\int_0^t S^L(t-\tau)|\phi|^2\phi d\tau\\
&=S^L(t)\left(\phi^+-i\int_{-\infty}^0 S^L(0-\tau)|\phi|^2\phi d\tau\right)\\
&\quad-i\int_0^t S^L(t-\tau)|\phi|^2\phi d\tau\\
&=S^L(t)\phi^++i\int_t^\infty S^L(t-\tau)|\phi|^2\phi d\tau.
\end{aligned}$$

We show how this last integral equation is solved for $\phi(x,t)$ using a fixed-point argument completely analogous to the one using in the proof of Theorem 3.3. By our global existence result and time reversibility, we may extend this solution ϕ, starting from data at time t_0, to all of $[0, \infty)$. It is now straighforward to verify that

$$\lim_{t \to \infty} \|\phi(t) - S^L(t)\phi^+\|_{H^s(\mathbb{R}^3)} = 0,$$

as we desired.

Appendix. Tools from Harmonic Analysis

In this Appendix we remind some well-known results about harmonic analysis and introduce some tools which have played a relevant role along the work.

A.1. The Fourier Transform

The Fourier transform is the very first instrument to work in harmonic analysis. It allows us, in a naif way, to understand a function as a trigonometric sum. The Fourier transform in a fixed point ξ give us the weight of the trigonometric term with frequence ξ in the sum. Here we give the definition of the Fourier transform and some basic properties which have been systematically used without mention.

Definition A.1. *Let $f \in L^1(\mathbb{R}^n)$ we define its Fourier transform by*

$$\mathscr{F}[f](\xi) = \widehat{f}(\xi) \equiv \int_{\mathbb{R}^n} f(x) e^{-2\pi i \langle x, \xi \rangle} dx, \quad \xi \in \mathbb{R}^n. \tag{A.1}$$

Theorem A.1 (Basic properties of the Fourier transform)**.** *Let $f \in L^1(\mathbb{R}^n)$ then*

1. *$f \longmapsto \widehat{f}$ maps the space $L^1(\mathbb{R}^n)$ into $L^\infty(\mathbb{R}^n)$, with*
$$\|\widehat{f}\|_\infty \leq \|f\|_1$$
2. *$\widehat{f}$ is continuous.*
3. *$\widehat{f}(\xi) \to 0$ when $|\xi| \to 0$ (Riemann-Lebesgue).*
4. *If we define $\tau_h f(x) \equiv f(x-h)$ for $h \in \mathbb{R}^n$, then*
$$\mathscr{F}[\tau_h f](\xi) = e^{-2\pi i \langle h, \xi \rangle} \widehat{f}(\xi),$$
$$\mathscr{F}[e^{-2\pi i \langle x, h \rangle} f](\xi) = \tau_{-h} \widehat{f}(\xi).$$
5. *If $\delta_a f(x) \equiv f(ax)$ for $a > 0$, then*
$$\mathscr{F}[\delta_a f](\xi) = a^{-n} \widehat{f}(a^{-1}\xi).$$
6. *Let $g \in L^1(\mathbb{R}^n)$ then*
$$\mathscr{F}[f * g](\xi) = \widehat{f}(\xi) \cdot \widehat{g}(\xi).$$

7. *Let $g \in L^1(\mathbb{R}^n)$ then*

$$\int_{\mathbb{R}^n} \widehat{f}(y)g(y)dy = \int_{\mathbb{R}^n} f(y)\widehat{g}(y)dy.$$

Proposition A.1. *Let $x_k f \in L^1(\mathbb{R}^n)$. Then $\widehat{f}$ is differentiable with respect to ξ_k and*

$$\frac{\partial \widehat{f}}{\partial \xi_k}(\xi) = \mathscr{F}[-2\pi i x_k f](\xi).$$

Definition A.2. *We'll say that $f \in L^p(\mathbb{R}^n)$ is differentiable in $L^p(\mathbb{R}^n)$ with respect to the variable x_k if there exists $g \in L^p(\mathbb{R}^n)$ such that*

$$\int_{\mathbb{R}^n} \left| \frac{f(x+he_k) - f(x)}{h} - g(x) \right| dx \to 0, \quad \text{when } h \to 0.$$

In this case, g is the partial k-derivative of f in the L^p-norm.

Proposition A.2. *If $f \in L^1(\mathbb{R}^n)$ and g is its partial k-derivative in the L^1-norm, then*

$$\widehat{g}(\xi) = 2\pi i \xi_k \widehat{f}(\xi).$$

Proposition A.3 (Inversion Formula). *Let $f, \widehat{f} \in L^1(\mathbb{R}^n)$, then*

$$f(x) = \int_{\mathbb{R}^n} e^{2\pi i \langle x, \xi \rangle} \widehat{f}(\xi) d\xi, \quad \text{a.e. } x \in \mathbb{R}^n.$$

Theorem A.2 (Plancherel). *Let $f \in L^1(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$ then $\widehat{f} \in L^2(\mathbb{R}^n)$ and*

$$\|\widehat{f}\|_2 = \|f\|_2.$$

Now we introduce the Fourier transform in the context of temperated distributions.

Definition A.3. *Let $(\nu, \beta) \in (\mathbb{Z}^+)^{2n}$. Denote the seminorm $[\cdot]_{(\nu,\beta)}$ by*

$$[f]_{(\nu,\beta)} = \|x^\nu \partial_x^\beta f\|_\infty.$$

We define the Schwartz space $\mathscr{S}(\mathbb{R}^n)$ by

$$\mathscr{S}(\mathbb{R}^n) = \{\varphi \in \mathcal{C}^\infty(\mathbb{R}^n) : [\varphi]_{(\nu,\beta)} < \infty, \quad \forall\, (\nu, \beta) \in (\mathbb{Z}^+)^{2n}\}.$$

Let $\{\varphi_j\} \subset \mathscr{S}(\mathbb{R}^n)$. We say that $\varphi_j \to 0$ if, for all pair $(\nu, \beta) \in (\mathbb{Z}^+)^{2n}$,

$$[\varphi_j]_{(\nu,\beta)} \to 0, \quad \text{as } j \to \infty.$$

The Schwartz space is a Frechet space with the topology given by the family of seminorms $[\cdot]_{(\nu,\beta)}$, $(\nu, \beta) \in (\mathbb{Z}^+)^{2n}$.

Theorem A.3. *The operator $\varphi \longmapsto \widehat{\varphi}$ is an isomorphism from $\mathscr{S}(\mathbb{R}^n)$ itself.*

Definition A.4. *We say that $\Psi : \mathscr{S}(\mathbb{R}^n) \to \mathbb{C}$ defines a temperated distribution, that is, $\Psi \in \mathscr{S}'(\mathbb{R}^n)$ if*

1. *Ψ is lineal.*
2. *Ψ is continuous, i.e., if $\{\varphi_j\} \subset \mathscr{S}(\mathbb{R}^n)$ and $\varphi_j \to 0$ as $j \to \infty$, then $\Psi(\varphi_j) \to 0$ as $j \to \infty$.*

Definition A.5. *Let $\{\Psi_j\} \subset \mathscr{S}'(\mathbb{R}^n)$. We say that $\Psi_j \to 0$ as $j \to \infty$ if, for all $\varphi \in \mathscr{S}'(\mathbb{R}^n)$,*

$$\Psi_j(\varphi) \to 0, \quad \text{as } j \to \infty.$$

Definition A.6. *Let $\Psi \in \mathscr{S}'(\mathbb{R}^n)$, we define its Fourier transform $\mathscr{F}[\Psi] = \widehat{\Psi} \in \mathscr{S}'(\mathbb{R}^n)$ by*

$$\widehat{\Psi}(\varphi) = \Psi(\widehat{\varphi}), \quad \text{for all } \varphi \in \mathscr{S}(\mathbb{R}^n).$$

Theorem A.4. *$\mathscr{F} : \Psi \longmapsto \widehat{\Psi}$ is an isomorphism from $\mathscr{S}'(\mathbb{R}^n)$ itself.*

A.2. The Calderón-Zygmund Theorem

The Calderón-Zygmund Theorem is a basic result in the theory of singular integrals. We shall use it in the next section of the Appendix in order to prove the Littlewood-Paley Theorem.

We begin by explain the *Calderón-Zygmund decomposition* in the following Lemma.

Lemma A.6. *Let $f \in L^1(\mathbb{R}^n)$ a non-negative function and $\lambda > 0$. Then there exists a sequence $\{Q_j\}$ of disjoint dyadic cubes such that*

1. $f(x) \leq \lambda$ *a.e.* $x \notin \bigcup_j Q_j$;
2. $|\bigcup_j Q_j| \leq \frac{1}{\lambda}\|f\|_1$;
3. $\lambda < \frac{1}{|Q_j|}\int_{Q_j} f \leq 2^n\lambda.$

Proof. Let $\mathscr{Q}_k$ the family of dyadic cubes in the grille $(2^{-k}\mathbb{Z})^n$. We define

$$E_k f(x) = \sum_{Q\in\mathscr{Q}_k}\left(\frac{1}{|Q|}\int_Q f\right)\chi_Q(x).$$

We also define the maximal dyadic function by

$$M_d f(x) = \sup_k |E_k f(x)|.$$

Now we may write

$$\{x : M_d f(x) > \lambda\} = \bigcup_k \Omega_k,$$

where

$$\Omega_k = \{x : E_k f(x) > \lambda, \text{ and } E_j f(x) \leq \lambda \text{ for } j < k\}.$$

The sets Ω_k are disjoint, and such of them may be writen as the union of cubes of the family $\mathscr{Q}_k$, i.e.,

$$\bigcup_k \Omega_k = \bigcup_j Q_j.$$

where these cubes Q_j are which we are looking for. Then,

$$\begin{aligned}|\{x : M_d f(x) > \lambda\}| = \sum_k |\Omega_k| &\leq \sum_k \frac{1}{\lambda}\int_{\Omega_k} E_k f \\ &= \frac{1}{\lambda}\sum_k \int_{\Omega_k} f \\ &\leq \frac{1}{\lambda}\|f\|_1.\end{aligned}$$

This proves part *2*. It is clear now that if f is continuous, then $\lim_{k\to\infty} E_k f(x) = f(x)$ for all $x \in \mathbb{R}^n$, and by density, if $f \in L^1(\mathbb{R}^n)$ then

$$\lim_{k\to\infty} E_k f(x) = f(x) \quad \textit{a.e.} \tag{A.2}$$

Note that if $x \notin \cup_j Q_j$ then $E_j f(x) \leq \lambda$ for all j, so by (A.2) it follows part *1*.

Finally, Q_j takes part in the decomposition if the mean of f over Q_j is greater than λ (which is the first inequality of *3*)., and the mean of f over the cubic $\widetilde{Q}_j \supset Q_j$, with twice the side of Q_j, is less or iqual to λ. Thus

$$\frac{1}{|Q_j|}\int_{Q_j} f \leq \frac{|\widetilde{Q}_j|}{|Q_j|}\frac{1}{|\widetilde{Q}_j|}\int_{\widetilde{Q}_j} f \leq 2^n\lambda.$$

□

Definition A.7. *A Calderón-Zygmund operator T is a linear operator*[1] *on $\mathbb{R}^n$ of the form*

$$Tf(x) = \int_{\mathbb{R}^n} K(x,y)f(y)dy$$

for some (possibly distributional, possibly matrix valued) kernel K which obeys the bounds

$$|K(x,y)| \lesssim \frac{1}{|x-y|^n} \tag{A.3}$$

and

$$|\nabla K(x,y)| \lesssim \frac{1}{|x-y|^{n+1}} \tag{A.4}$$

for all $x \neq y$. We also require that T be bounded on L^2.

We know show the following version of the Calderón-Zygmund theorem.

Theorem A.5 (Calderón-Zygmund)**.** *If T is a Calderón-Zygmund operator, then T is bounded on all $L^p(\mathbb{R}^n)$, $1 < p < \infty$.*

Proof. It suffices by duality to check the case $1 < p < 2$, since the class of Calderón-Zygmund operators is self-adjoint. By the Marcinkiewicz interpolation theorem it suffices to show that T is week-type $(1,1)$, i.e. that

$$|\{|Tf| > \lambda\}| \lesssim \frac{\|f\|_1}{\lambda}. \tag{A.5}$$

To avoid irrelevant technicalities we shall only prove the one-dimensional case $n = 1$. Using Lemma A.6, we obtain a sequence of disjoint intervals $\{I_j\}$ such that

$$f(x) \leq \lambda \text{ a.e. } x \notin \Omega = \bigcup_j I_j; \tag{A.6}$$

$$|\Omega| \leq \frac{1}{\lambda}\|f\|_1; \tag{A.7}$$

$$\lambda < \frac{1}{|I_j|}\int_{I_j} f \leq 2\lambda. \tag{A.8}$$

Now we split f as a sum of two functions g and b given by

$$g(x) = \begin{cases} f(x) & \text{if } x \notin \Omega, \\ \dfrac{1}{|I_j|}\displaystyle\int_{I_j} f & \text{if } x \in I_j. \end{cases}$$

[1] There are more general notions of a Calderón-Zygmund kernel, but we don't need them in this work.

$$b(x) = \sum_j b_j(x) \text{ with } b_j(x) = \left(f(x) - \frac{1}{|I_j|} \int_{I_j} f \right) \chi_{I_j}(x).$$

We shall estimate the left-hand side of (A.5) by

$$|\{Tg \geq \lambda/2\}| + |\{|\sum_j Tb_j| \geq \lambda/2\}|.$$

We can estimate the first term using Chebyshev's inequality by

$$|\{Tg \geq \lambda/2\}| \lesssim \lambda^{-2} \|Tg\|_2^2 \lesssim \lambda^{-2} \|g\|_2^2 \lesssim \lambda^{-2} \|g\|_1 \|g\|_\infty.$$

From (A.8) we see that $\|b_j\|_1 \lesssim \lambda |I_j|$. Since $f = g + b$, we thus see from (A.7) that $\|g\|_1 \lesssim \|f\|_1$. Also, from (A.8) and the construction of the I_j we see that $\|g\|_\infty \lesssim \lambda$. Thus this term is acceptable.

Now we control the second term $|\{|\sum_j Tb_j| \geq \lambda/2\}|$. The function b_j is supported in I_j and has mean zero. We call y_j for the center of the interval I_j. By (A.4) and the mean value theorem we have the estimates

$$\begin{aligned}
|Tb_j(x)| &= |\int_{I_j} b_j(y) K(x,y) dy| \\
&= |\int_{I_j} b_j(y)(K(x,y) - K(x,y_j)) dy| \\
&\lesssim \int_{I_j} |b_j| \frac{|y - y_j|}{|x - y_j|^2} dy \\
&\lesssim |I_j| \|b_j\|_1 \frac{1}{\operatorname{dist}(x, I_j)^2} \\
&\lesssim \lambda |I_j|^2 \frac{1}{\operatorname{dist}(x, I_j)^2},
\end{aligned}$$

whenever $x \notin 2I_j$. Thus, outside the exceptional set $\Omega^* = \bigcup_j 2I_j$ we have

$$\|Tb_j\|_{L^1(\mathbb{R} \setminus \Omega^*)} \lesssim \lambda |I_j|.$$

Then, by using (A.7) we have

$$\|\sum_j Tb_j\|_{L^1(\mathbb{R} \setminus \Omega^*)} \lesssim \|f\|_1$$

and so this part is also acceptable by Chebyshev's inequality:

$$\begin{aligned}
|\{|\sum_j Tb_j| \geq \lambda/2\}| &\leq |\Omega^*| + |\{x \notin \Omega^* : |\sum_j Tb_j(x)| \geq \lambda/2\}| \\
&\lesssim \lambda^{-1} \|f\|_1 + \lambda^{-1} \|\sum_j Tb_j\|_1 \\
&\lesssim \lambda^{-1} \|f\|_1.
\end{aligned}$$

This concludes the proof. □

A.3. The Hardy-Littlewood-Sobolev Theorem

We begin by defining the *Hardy-Littlewood maximal function.*

Definition A.8. *For a given $f \in L^1_{loc}(\mathbb{R}^n)$, we define $\mathcal{M}f(x)$, the Hardy-Littlewood maximal function associated to f, as*

$$\begin{aligned}\mathcal{M}f(x) &= \sup_{r>0} \frac{1}{|B_r(x)|} \int_{B_r(x)} |f(y)|dy \\ &= \sup_{r>0} \left(|f| * \frac{1}{|B_r(0)|}\chi_{B_r(0)} \right)(x).\end{aligned}$$

It's easy to verify the next properties of $\mathcal{M}$:

Proposition A.4. *Let $f, g \in L^1_{loc}(\mathbb{R}^n)$, then*

1. *$\mathcal{M}$ define a sublinear operator, i.e.,*

$$|\mathcal{M}(f+g)(x)| \leq |\mathcal{M}f(x)| + |\mathcal{M}g(x)|, \quad a.e.\ x \in \mathbb{R}^n.$$

2. *If $f \in L^\infty(\mathbb{R}^n)$, then*

$$\|\mathcal{M}f\|_\infty \leq \|f\|_\infty.$$

The following is a technical lemma needed in the proofs below.

Lemma A.7 (Vitali's covering lemma; [13], page 33)**.** *Let $E \subset \mathbb{R}^n$ be a measurable set such that $E \subset \bigcup_\alpha B_{r_\alpha}(x_\alpha)$, with the family of open balls $\{B_{r_\alpha}(x_\alpha)\}_\alpha$ satisfying $\sup_\alpha r_\alpha = c_0 < \infty$. Then there exists a subfamily $\{B_{r_j}(x_j)\}_j$ disjoint and numerable such that*

$$|E| \leq 5^n \sum_{j=1}^{\infty} |B_{r_j}(x_j)|.$$

Theorem A.6 (Hardy-Littlewood)**.** *Let $1 < p \leq \infty$. Then $\mathcal{M}$ is a quasilinear operator of type (p,p), i.e.*

$$\|\mathcal{M}f\|_p \lesssim \|f\|_p,$$

for all $f \in L^p(\mathbb{R}^n)$, where the implicit constant depends on p.

Proof. Since $\|\mathcal{M}f\|_\infty \leq \|f\|_\infty$, by the Marcinkiewicz interpolation theorem it suffices to show that $\mathcal{M}$ is of weak type $(1,1)$, that is,

$$\sup_{\lambda>0} \lambda \cdot m(\lambda, \mathcal{M}f) \lesssim \|f\|_1,$$

where $m(\lambda, \mathcal{M}f) = |\{x \in \mathbb{R}^n : |\mathcal{M}f(x)| > \lambda\}|$. We denote in general

$$m(\lambda, f) = |\{x \in \mathbb{R}^n : |f(x)| > \lambda\}| = |E_f^\lambda|.$$

First, by Tchebyshev inequality,

$$m(\lambda, f) \leq \lambda^{-p} \|f\|_p^p.$$

Now we define $E^\lambda_{\mathcal{M}f} = \{x \in \mathbb{R}^n : \mathcal{M}f(x) > \lambda\}$. Thus, if $x \in E^\lambda_f$, then there exists $B_{r_x}(x)$ such that

$$\int_{B_{r_x}(x)} |f(y)|dy > \lambda |B_{r_x}(x)|.$$

Clearly, we have that

$$E^{\lambda}_{\mathcal{M}f} \subset \bigcup_{x \in E^{\lambda}_{\mathcal{M}f}} B_{r_x}(x).$$

By the Vitali's covering lemma, there exists $\{B_{r_{x_j}}(x_j)\}$ disjoint such that

$$|E^{\lambda}_{\mathcal{M}f}| \leq 5^n \sum_{j=1}^{\infty} |B_{r_{x_j}}(x_j)| \leq 5^n \lambda^{-1} \sum_{j=1}^{\infty} \int_{B_{r_{x_j}}(x_j)} |f(y)| dy \leq 5^n \lambda^{-1} \|f\|_1.$$

This completes the proof. □

Proposition A.5. *Let $\varphi \in L^1(\mathbb{R}^n)$ be a radial, positive, and non increasing function of $r = |x| \in [0, \infty)$. Then*

$$\sup_{t>0} |\varphi_t * f(x)| = \sup_{t>0} \left| \int_{\mathbb{R}^n} \frac{\varphi(t^{-1}(x-y))}{t^n} f(y) dy \right| \leq \|\varphi\|_1 \mathcal{M}f(x). \tag{A.9}$$

Proof. We first assume that φ is a simple function

$$\varphi(x) = \sum_k a_k \chi_{B_{r_k}(0)}(x),$$

with $a_k > 0$. Hence,

$$\varphi * f(x) = \sum_k a_k |B_{r_k}(0)| \frac{1}{|B_{r_k}(0)|} \chi_{B_{r_k}(0)} * f(x) \leq \|\varphi\|_1 \mathcal{M}f(x).$$

In the general case in which $\varphi \in L^1(\mathbb{R}^n)$, take $\varphi_n \to \varphi$ a sequence of simple functions satisfying the hypotheses. Since dilatations of φ satisfy the same hypotheses and preserve the L^1 norm we can pass to the limit and obtain the desired result. □

We now show the result that gives name to this section.

Definition A.9. *Let $0 < \alpha < n$. The Riesz potential of order α, denoted by I_α is defined as*

$$I_\alpha f(x) = c_\alpha \int_{\mathbb{R}^n} \frac{f(y)}{|x-y|^{n-\alpha}} dy = \kappa_\alpha * f(x). \tag{A.10}$$

Theorem A.7 (Hardy-Littlewood-Sobolev)**.** *Let $o < \alpha < n$, and $1 \leq p < q < \infty$ such that*

$$\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{n}.$$

Then

1. *If $f \in L^p(\mathbb{R}^n)$ then (A.10) is absolutely convergent a.e. $x \in \mathbb{R}^n$.*
2. *If $p > 1$, I_α is of type (p, q), i.e.,*

$$\|I_\alpha f\|_q \leq c_{p,\alpha,n} \|f\|_p.$$

Proof. We split the kernel $\kappa_\alpha(x) = \kappa_\alpha^0(x) + \kappa_\alpha^\infty(x)$, where

$$\kappa_\alpha^0(x) = \begin{cases} \kappa_\alpha(x) & \text{if } |x| \leq \epsilon, \\ 0 & \text{if } |x| > \epsilon, \end{cases}$$

with ϵ to be choseen early. We have

$$|I\alpha f(x)| \leq |\kappa_\alpha^0 * f(x)| + |\kappa_\alpha^\infty * f(x)| = I + II.$$

The integral I represents the convolution of a function $\kappa_\alpha^0 \in L^1(\mathbb{R}^n)$ with $f \in L^p(\mathbb{R}^n)$. The integral II is the convolution of a function $f \in L^p(\mathbb{R}^n)$, II also converges with $\kappa_\alpha^\infty \in L^{p'}(\mathbb{R}^n)$. Therefore both integrals converge absolutely.

Also, using

$$\int_{|y|<\epsilon} \frac{dy}{|u|^{n-\alpha}} = c_n \int_0^\epsilon \frac{r^{n-1}}{r^{n-\alpha}} dr = c_{n,\alpha}\epsilon^\alpha$$

and (A.9), we infer that

$$I \leq \epsilon^\alpha \left(\frac{1}{\epsilon^\alpha} \psi_{\{|y/\epsilon|<1\}}(y) \frac{1}{|y|^{n-\alpha}} * |f| \right)(x) \leq c_{\alpha,n}\epsilon^\alpha \mathcal{M}f(x).$$

On the other hand, by Hölder's inequality,

$$\begin{aligned} II &\leq c_{\alpha,n}\|f\|_p \left(\int_{|y|\geq\epsilon} \frac{1}{|y|^{(n-\alpha)p'}} dy \right)^{1/p'} \\ &= c_{\alpha,n}\|f\|_p \left(\int_\epsilon^\infty \frac{r^{n-1}}{r^{(n-\alpha)p'}} dr \right)^{1/p'} \\ &= c_{\alpha,n}\epsilon^{n/p'-n+\alpha}\|f\|_p. \end{aligned}$$

Now we fix $\epsilon = \epsilon(x)$ such that

$$c\epsilon^\alpha \mathcal{M}f(x) = c\epsilon^{n/p'-n+\alpha}\|f\|_p.$$

Using $n/p' - n = -n/p$, this is equivalent to

$$c\mathcal{M}f(x) = c\epsilon^{-n/p}\|f\|_p.$$

Combining these expressions, we obtain

$$\begin{aligned} |I_\alpha f(x)| &\leq c \left(\|f\|_p (\mathcal{M}f(x))^{-1} \right)^{\alpha p/n} \mathcal{M}f(x) \\ &= c\|f\|_p^{\alpha p/n} (\mathcal{M}f(x))^{1-\alpha p/n} \\ &= c\|f\|_p^\theta (\mathcal{M}f(x))^{1-\theta}, \end{aligned} \tag{A.11}$$

where $\theta = \alpha p/n \in (0,1)$. Finally, taking L^q-norm in (A.11) and using Theorem A.6 we conclude

$$\|I_\alpha f\|_q \leq c\|f\|_p^\theta \|(\mathcal{M}f)^{1-\theta}\|_q = c\|f\|_p^\theta \|\mathcal{M}f\|_{(1-\theta)q}^{1-\theta} \leq c\|f\|_p,$$

since $(1-\theta)q = (1-\alpha p/n)q = p$.

$\square$

A.4. Littlewood-Paley Theory and Multipliers

The *Littlewood-Paley decomposition* is a very basic way to carve the phase plane. There is a certain amount of flexibility in how one sets up the Littlewood-Paley decomposition on $\mathbb{R}^n$, but one standard way is as follows. Let $\phi(\xi)$ be a real radial bump function such that

$$\begin{cases} \text{supp } \phi \subset \{\xi \in \mathbb{R}^n : \ |\xi| \leq 2\}, \\ \phi(\xi) \equiv 1 \text{ in } \{\xi \in \mathbb{R}^n : \ |\xi| \leq 1\}. \end{cases}$$

Let $\psi(\xi)$ be the function

$$\psi(\xi) = \phi(\xi) - \phi(2\xi).$$

Thus ψ is a bump function supported on the annulus $\{1/2 \leq \xi \leq 2\}$. By construction we have

$$\sum_{k\in\mathbb{Z}} \psi(\xi/2^k) = 1,$$

for all $\xi \neq 0$.

We define the *Littlewood-Paley projection operators* P_k, $P_{\leq k}$ by

$$\widehat{P_k f}(\xi) = \psi(\xi/2^k)\widehat{f}(\xi),$$

$$\widehat{P_{\leq k} f}(\xi) = \phi(\xi/2^k)\widehat{f}(\xi).$$

Observe that $P_k = P_{\leq k} - P_{\leq k-1}$. Also, if f is an L^2 function then

$$\|P_{\leq k} f\|_2 = \|\widehat{P_{\leq k} f}\|_2 = \|\phi(\cdot/2^k)\widehat{f}\|_2 \to 0, \quad \text{as } k \to -\infty$$

and

$$\|P_{\leq k} f - f\|_2 = \|\phi(\cdot/2^k)\widehat{f} - \widehat{f}\|_2 \to 0, \quad \text{as } k \to \infty.$$

Now we define the vector-valued function

$$Sf \equiv (P_k f)_{k\in\mathbb{Z}}; \quad |Sf| \equiv \left(\sum_k |P_k f|^2\right)^{1/2}.$$

Here we can give the *Littlewood-Paley inequality.*

Theorem A.8 (Littlewood-Paley inequality)**.** *For any* $1 < p < \infty$*, we have*

$$\|Sf\|_p = \|(\sum_k |P_k f|^2)^{1/2}\|_p \sim \|f\|_p$$

with the implicit constant depending on p*.*

Proof. We observe that S is a vector-valued Calderón-Zygmund operator with vector-valued kernel

$$K(x,y) = (2^{nk}\widehat{\psi}(2^k(x-y)))_{k\in\mathbb{Z}},$$

the decay properties (A.3) and (A.4) yields since $\widehat{\psi}$ is a Schwartz function. The L^2 boundness come from Plancherel theorem, that allows us to write

$$\|f\|_2 \sim \|(\sum_k |P_k f|^2)^{1/2}\|_2.$$

Then, from Theorem A.5 we have

$$\|Sf\|_p \lesssim \|f\|_p$$

for all $1 < p < \infty$. We observe now that

$$\begin{array}{rccc} S: & L^p & \longrightarrow & L^p(l^2) \\ & f & \longmapsto & (P_k f)_{k\in\mathbb{Z}}; \end{array}$$

while

$$\begin{array}{rccc} S^*: & L^q(l^2) & \longrightarrow & L^q \\ & (f_k)_{k\in\mathbb{Z}} & \longmapsto & \sum_k P_k f_k. \end{array}$$

where $\frac{1}{p} + \frac{1}{q} = 1$. It's clear that S and S^* are adjoint operators. Then, by duality we also have

$$\|S^* f\|_p \lesssim \|f\|_p,$$

for all $1 < p < \infty$. By untangling this, we see that

$$\|\sum_k P_k f_k\|_p \lesssim \|(\sum_k |f_k|^2)^{1/2}\|_p.$$

Similarly we have

$$\|\sum_k \widetilde{P}_k f_k\|_p \lesssim \|(\sum_k |f_k|^2)^{1/2}\|_p,$$

where $\widetilde{P}_k = P_{\le k+2} - P_{\le k-2}$.

We apply this with $f_k \equiv P_k f$. Since $\widetilde{P}_k P_k = P_k$, we obtain

$$\|f\|_p \lesssim \|(\sum_k |P_k f|^2)^{1/2}\|_p$$

which is the other half of the theorem. □

Remark A.1. Some easy consequences of the above theory are as follows. If for each k, the operator $\widetilde{P}_k$ is given by a bump function adapted to the annulus $\{|\xi| \sim 2^k\}$, then

$$\|(\sum_k |\widetilde{P}_k f|^2)^{1/2}\|_p \lesssim \|f\|_p \tag{A.12}$$

for all $1 < p < \infty$. A dual statement is that

$$\|\sum_k \widetilde{P}_k f_k\|_p \lesssim \|(\sum_k |f_k|^2)^{1/2}\|_p \tag{A.13}$$

for all $1 < p < \infty$ and arbitrary functions f_k.

The following theorem is consequence of the Litlewood-Paley theory.

Theorem A.9 (Hörmander, [9], page 162)**.** *Let $m(\xi) \in \mathcal{C}^k(\mathbb{R}^n)$ with $k = \lfloor \frac{n}{2} \rfloor + 1$ be a multiplier such that*

$$|D^\beta m(\xi)| \lesssim |\xi|^{-|\beta|}, \quad \text{for all } |\beta| \le k. \tag{A.14}$$

Let T_m be the Fourier multiplier with symbol m:

$$\widehat{T_m f}(\xi) = m(\xi) f(\xi).$$

Then, T_m is bounded on L^p for all $1 < p < \infty$.

Proof. We have

$$T_m = \sum_{k,k'} P_k T_m P_{k'} = \sum_k \widetilde{P}_k Q_k,$$

where $\widetilde{P}_k \equiv P_k T_m$ and $Q_k \equiv \sum_{k-2<k'<k+2} P_{k'}$.

It can be seen, using the condition (A.14), that the operator

$$\begin{aligned} \widetilde{S}: \quad L^2 &\longrightarrow \quad L^2(l^2) \\ f &\longmapsto \quad (\widetilde{P}_k f)_{k\in\mathbb{Z}} \end{aligned}$$

is a vector-valued Calderón-Zygmund, and then, by (A.13) we have

$$\|\sum_k \widetilde{P}_k f_k\|_p \lesssim \|(\sum_k |f_k|^2)^{1/2}\|_p.$$

Then, taking $f_k = Q_k f$ and using (A.12) we conclude that

$$\|(\sum_k |Q_k f|^2)^{1/2}\|_p \lesssim \|f\|_p.$$

□

Finally we enunciate a multiplier-theorem that we have used in the work.

Theorem A.10 (Coifman-Meyer, [6]). *Consider an infinitely differentiable symbol $\sigma : \mathbb{R}^{nk} \to \mathbb{C}$ so for all $\alpha \in \mathbb{N}^{nk}$ and all $\xi = (\xi_1, \dots, \xi_k) \in \mathbb{R}^{nk}$, there is a constant $c(\alpha)$ with*

$$|\partial_\xi^\alpha \sigma(\xi)| \leq c(\alpha)(1 + |\xi|)^{-|\alpha|}. \tag{A.15}$$

Define the multilinear operator Λ by

$$\begin{aligned} &[\Lambda(f_1, \dots, f_k)](x) = \\ &\qquad \int_{\mathbb{R}^{nk}} e^{ix(\xi_1+\cdots+\xi_k)} \sigma(\xi_1, \dots, \xi_k) \widehat{f}_1(\xi_1) \cdots \widehat{f}_k(\xi_k) d\xi_1 \cdots d\xi_k. \end{aligned} \tag{A.16}$$

Suppose $p_j \in (1, \infty), j = 1, \dots, k$, are such that

$$\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2} + \cdots + \frac{1}{p_k} \leq 1.$$

Then there is a constant $C = C(p_j, n, k, c(\alpha))$ so that for all Schwartz class functions $f_1, \dots, f_k$,

$$\|\Lambda(f_1, \dots, f_k)\|_{L^p(\mathbb{R}^n)} \leq C \|f_1\|_{L^{p_1}(\mathbb{R}^n)} \cdots \|f_k\|_{L^{p_k}(\mathbb{R}^n)}. \tag{A.17}$$

A.5. Sobolev Spaces and Fractional Leibniz Rules

Littlewood-Paley theory is especially good for dealing with spaces which combine L^p type norms with derivatives. The most commonly used family of such spaces are the *Sobolev spaces* $W^{s,p}$. In the case $p = 2$ these spaces are denoted by H^s. The $W^{s,p}$ are defined for $s \in \mathbb{R}$ and $1 < p < \infty$. When $s \geq 0$ is a non-negative integer, the spaces can be easily defined by

$$\|f\|_{W^{s,p}} \equiv \sum_{j=0}^{s} \|\nabla^j f\|_p$$

When $s = 0$ we have $W^{0,p} = L^p$.

Before we define these spaces for non-integer s, we first try to rewrite the above definition in terms of Littlewood-Paley operators P_k rather than derivatives ∇. We observe that

Lemma A.8. *For any $j \geq 0$ and $1 < p < \infty$, we have*

$$\|\nabla^j f\|_p \sim \|(\sum_k |2^{jk} P_k f|^2)^{1/2}\|_p.$$

From this lemma we easily see that

$$\|f\|_{W^{s,p}} \sim \|(\sum_k |(1 + 2^k)^s P_k f|^2)^{1/2}\|_p.$$

Note that the right-hand side makes sense for all s, not just integer s. Because of this, it is natural to use the right-hand side to define the Sobolev space $W^{s,p}$ for all s.

Fractional Sobolev spaces are related to the notion of *fractional derivatives*. Recall that

$$\widehat{\nabla f}(\xi) = 2\pi i \xi \widehat{f}(\xi).$$

Because of this, it is natural to define the operator $|\nabla|$ by

$$\widehat{|\nabla| f}(\xi) = 2\pi |\xi| \widehat{f}(\xi),$$

and more generally $|\nabla|^s$ for arbitrary $s \in \mathbb{R}$ by

$$\widehat{|\nabla|^s f}(\xi) = (2\pi|\xi|)^s \widehat{f}(\xi).$$

We define the homogenous Sobolev spaces by

$$\|f\|_{\dot{W}^{s,p}} \equiv \||\nabla|^s f\|_p,$$

and in the case $p = 2$ we denote them by $\dot{H}^s$.

One can connect the Sobolev spaces of a given integrability exponent p more tightly by introducing modified fractional differentiation operators $\langle\nabla\rangle^s$ by

$$\widehat{\langle\nabla\rangle^s f}(\xi) \equiv \langle 2\pi\xi\rangle^s \widehat{f}(\xi),$$

where the *Japanese bracket* $\langle x\rangle$ is defined by $\langle x\rangle \equiv (1 + |x|^2)^{1/2}$. It is easy to verify using Littlewood-Paley theory that one has the estimates

$$\|f\|_{W^{s,p}} \sim \|\langle\nabla\rangle^s f\|_p.$$

Now we enunciate a generalization of the well-known Sobolev embedding theorem:

Theorem A.11 (Sobolev embedding)**.** *Let $1 < p < q < \infty$ and $s' < s$ be such that*

$$\frac{1}{p} - \frac{s - s'}{n} = \frac{1}{q}.$$

Then we have

$$\|f\|_{W^{q,s}(\mathbb{R}^n)} \lesssim \|f\|_{W^{p,s'}(\mathbb{R}^n)}$$

for all functions f for which the right-hand side is finite. (The implicit constant depends on p, q, n, s, s').

The following proposition has been used in the work. We include it without proof.

Proposition A.6 (Hardy's inequality; [19], page 334). *If $0 \leq s < n/2$ then*

$$\big\| |x|^{-s} f \big\|_{L^2(\mathbb{R}^n)} \leq C_{s,n} \|f\|_{\dot{H}^s(\mathbb{R}^n)}. \tag{A.18}$$

In the last part of this appendix, we show the called *fractional Leibniz rule*. References about these topics are [5], [12], and [20].

Proposition A.7 (Fractional Leibniz Rule). *Let $s \in (0,1)$, $1 < r$, p_1, p_2, q_1, $q_2 < \infty$, and suppose*

$$\frac{1}{r} = \frac{1}{p_1} + \frac{1}{q_1} = \frac{1}{p_2} + \frac{1}{q_2}.$$

Suppose that $f \in L^{p_1}$, $\langle \nabla \rangle^s f \in L^{p_2}$, $g \in L^{q_2}$, $\langle \nabla \rangle^s g \in L^{q_1}$. Then $\langle \nabla \rangle^s (fg) \in L^r$ and

$$\|\langle \nabla \rangle^s (fg)\|_r \lesssim \|f\|_{p_1} \|\langle \nabla \rangle^s g\|_{q_1} + \|\langle \nabla \rangle^s f\|_{p_2} \|g\|_{q_2} \tag{A.19}$$

Proof. We consider the operators P_k given by

$$\widehat{P_k f}(\xi) = \psi(\xi/2^k) \widehat{f}(\xi),$$

with ψ a bump function supported in the annulus $\{1/2 \leq |\xi| \leq 2\}$. Recall that

$$\sum_{k \in \mathbb{Z}} \psi(\xi/2^k) = 1,$$

for all $\xi \neq 0$. We also have seen that, by Littlewood-Paley theory, we have

$$\|\langle \nabla \rangle^s h\|_p \sim \|(\sum_k |(1+2^k)^s P_k h|^2)^{1/2}\|_p,$$

for all $h \in W^{s,p}$.

Define in the same way operators $\widetilde{P}_k$ from a bump function $\widetilde{\psi}$ identically one on $\{1/4 \leq |\xi| \leq 4\}$ and supported on $\{1/8 < |\xi| < 8\}$. Consider

$$Q_j f = \sum_{k \leq j-3} P_k f.$$

Then we have

$$P_k g \cdot Q_k f = \widetilde{P}_k (P_k g \cdot Q_k f)$$

for all f, g. Write

$$\begin{aligned} fg &= \sum_k P_k g \cdot Q_k f + \sum_k P_k f \cdot Q_k g + \sum_{|i-j| \leq 2} P_i f \cdot P_j g \\ &= \sum_k \widetilde{P}_k (P_k g \cdot Q_k f) + \sum_k \widetilde{P}_k (P_k f \cdot Q_k g) \\ &\quad + \sum_{|i-j| \leq 2} \widetilde{P}_k (P_i f \cdot P_j g). \end{aligned}$$

For the first term

$$
\begin{aligned}
\|(\sum_k |(1+2^k)^s \widetilde{P}_k(P_k g \cdot Q_k f)|^2)^{1/2}\|_r \\
&\lesssim \|(\sum_k (M((1+2^k)^s P_k g \cdot Q_k f))^2)^{1/2}\|_r \\
&\lesssim \|(\sum_k (1+2^k)^{2s} |P_k g|^2 \cdot (Mf)^2)^{1/2}\|_r \\
&\lesssim \|Mf\|_{p_1} \|(\sum_k (1+2^k)^{2s} |P_k g|^2)^{1/2}\|_{q_1} \\
&\lesssim \|f\|_{p_1} \|\langle \nabla \rangle^s g\|_{q_1}.
\end{aligned}
$$

The second term is the same, but with the roles of f and g reversed. For the third, when $|i-j| \leq 2$, $\widetilde{P}_k(P_i f \cdot P_j g) \equiv 0$ unless $k \leq \max(i,j)+4$. Thus

$$
\begin{aligned}
(\sum_k (1+2^k)^{2s} |\widetilde{P}_k(\sum_{|i-j|\leq 2} P_i f \cdot P_j g)|^2)^{1/2} \\
&\lesssim (\sum_k (1+2^k)^{2s} |\widetilde{P}_k \sum_{\substack{|i-j|\leq 2 \\ \max(i,j)\geq k-4}} (P_i f \cdot P_j g)|^2)^{1/2} \\
&\lesssim \sum_{l\geq -6} \sum_{|m|\leq 2} (\sum_j (1+2^{j-l})^{2s} |\widetilde{P}_{j-l}(P_{j-m} f \cdot P_j g)|^2)^{1/2} \\
&\lesssim \sum_{l\geq -6} \sum_{|m|\leq 2} 2^{-ls} (\sum_j (1+2^{j})^{2s} |\widetilde{P}_{j-l}(P_{j-m} f \cdot P_j g)|^2)^{1/2}.
\end{aligned}
$$

The L^r norm is then estimated as above. $\square$

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