

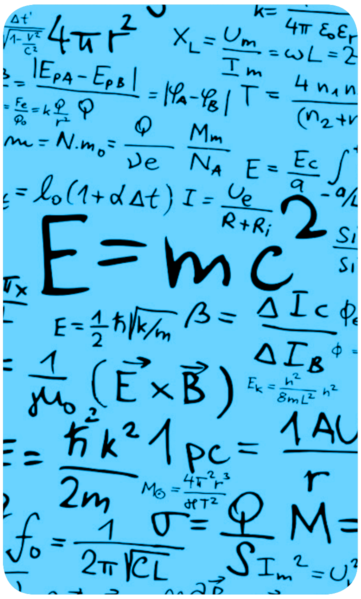

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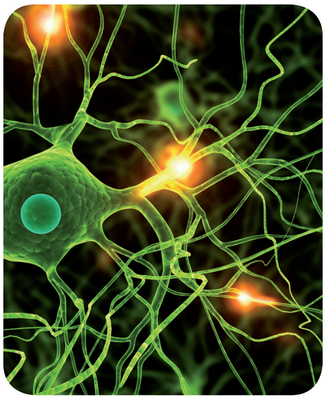
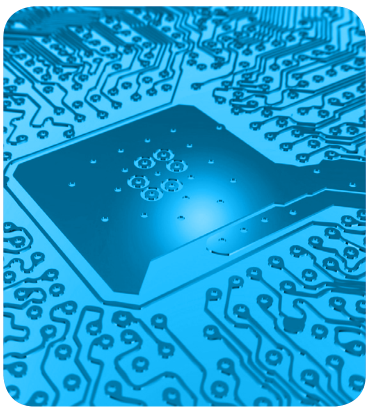


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Torelli Theorem for the parabolic Deligne-Hitchin moduli space over a curve

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Abstract

In this master thesis we describe a parabolic version of the Deligne-Hitchin moduli space for $\mathrm{SL}(r,\mathbb{C})$-bundles over punctured Riemann surfaces and we prove a new Torelli theorem for this space. After reviewing some of the basic concepts about moduli spaces of vector bundles and Higgs bundles, the formalism of parabolic structures is given in order to deal with noncompact curves.

Parabolic vector bundles and parabolic Higgs bundles are described, and we present the relations of these objects with parabolic connections and filtered local systems. Torelli theorems are stated for all the previously described moduli spaces and the Deligne-Hitchin moduli space is built from the Hodge moduli space.

Finally, the generalization of the Hodge moduli space and the Deligne-Hitchin moduli space for the parabolic case is described. An alternative proof of the Torelli theorem for the parabolic Higgs moduli space is given, and the Torelli theorem for both the Hodge moduli space and the Deligne-Hitchin moduli space is proved.

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Key words and phrases. Vector bundle, moduli space, parabolic Deligne-Hitchin moduli, parabolic Higgs bundle, Torelli.

Contents

Introduction

The Jacobian variety associated to an algebraic curve is a very important object in algebraic geometry. It has been deeply studied for decades and a great number of theorems have been proven about it. One of the main theorems referring the Jacobian variety is the Torelli theorem. It states that the isomorphism class of a polarized Jacobian variety of an algebraic curve determines the isomorphism class of the curve.

The Jacobian variety parametrizes line bundles over the curve, i.e., vector bundles of rank one. Thus, a natural generalization is trying to build a variety parameterizing vector bundles of an arbitrary fixed rank in the sense of a moduli space. This generalization is not straightforward, as it was proved that, in general, there does not exist a moduli space that represents the whole set of isomorphism classes of vector bundles of a certain fixed rank.

Mumford [Mum62] developed the notion of stability of a vector bundle and proved that there exist a moduli space of stable vector bundles over an algebraic curve with fixed rank and degree. On the other hand, Narasimhan and Seshadri [NS65] studied the relation between vector bundles over a curve and representations of its fundamental group. Surprisingly, they found that the condition for a vector bundle to correspond to an irreducible unitary representation of the fundamental group was exactly the one given by Mumford's stability.

This correspondence led up to questioning about the kind of geometric objects that would correspond to a general linear representation of the fundamental group of a curve. Higgs bundles, defined by Hitchin while studying self-duality equations on Riemann surfaces [Hit87a], were described by Simpson as the counterpart of a non-irreducible representation of the fundamental group of the curve [Sim92].

Similarly to the Jacobian variety, a Toerelli theorem for the moduli space of semistable vector bundles was developed [NR75] and, after that, another Torelli theorem was proven for the moduli space of semistable Higgs bundles over the curve [BG03].

The relations between Higgs bundles, connections and representations of the fundamental group [Sim95] allows us to define the Deligne-Hitchin moduli space both as a space that "glues" together the previously stated moduli spaces and as the twistor space of the moduli space of semistable Higgs bundles. Thus, the geometry of the Deligne-Hitchin moduli space of a curve is of great interest, as describes globally the interactions between Higgs bundles, connections and representations of the fundamental group of a Riemann surface. A Torelli theorem for the Deligne-Hitchin moduli space for compact connected Riemann surfaces was stated in [BGHL09].

All the previously described objects are meant not to have singularities over the curve. The corresponding Torelli theorems are stated for compact Riemann surfaces, thus omitting the study of the singular scenario. Parabolic vector bundles appear as a tool to analyze singularities at a certain set of punctures over a curve. Parabolic Higgs bundles and parabolic connections are built as a singular generalization of the corresponding classical notions that are allowed to have "logarithmic" singularities over certain prescribed punctures over a curve. Torelli theorems are stated for the moduli spaces of parabolic vector bundles [BdBnB01] and parabolic Higgs bundles [GL11] under certain conditions.

The objective of this master thesis is to build a parabolic counterpart for the Deligne-Hitchin moduli space and prove an original Torelli theorem for this space. In the process, a Torelli theorem for the parabolic Hodge moduli space will result and a new alternative demonstration of the Torelli theorem for the moduli space of semistable parabolic Higgs bundles will be given.

In the first chapter we will review some of the basic properties of vector bundles that will be used through the rest of this work. We will specially focus on the study of the vector bundles from the formalism of sheaf theory.

The second chapter will be dedicated to the formal definition of a moduli space. Fine and coarse moduli spaces will be defined precisely and some of their main properties will be stated. We will define the Jacobian problem as a core example of a moduli problem, and we will outline the proof of the construction of the corresponding fine moduli space. As the moduli space of vector bundles represent a natural generalization for the Jacobian over a curve, it will be also studied in this chapter. We will present the Mumford stability and analyze the principal characteristics of a semistable vector bundle.

Chapters three to six will be devoted to the construction of the different kinds of geometric objects that will be needed to define the parabolic Deligne-Hitchin moduli space. In chapter three, we will define the notion of Higgs bundle and the generalization of Mumford stability for these structures. We will also study the Hitchin map as an important tool in order to analyze the structure of the moduli space of semistable Higgs bundles. Chapter four will be focused in describing the formalism of parabolic vector bundles as a tool to treat singularities over a punctured curve. In chapter five, we will use this formalism to generalize Higgs bundles, allowing singular Higgs fields with logarithmic singularities over a punctured Riemann surface.

The sixth chapter describes both the filtered objects corresponding to connections and representations of the fundamental group within the parabolic formalism and the Simpson correspondences that exist between these families of parabolic objects and parabolic Higgs bundles.

In chapter seven, we will give an historical and logical overview to the development of the different versions of the Torelli theorem proved over the moduli spaces described in the previous chapters, from the classical Torelli theorem for the polarized Jacobian variety to the Torelli theorem for the moduli space of semistable parabolic Higgs bundles. We will also present the main ideas leading up to the proof of some of these theorems.

Chapter eight describes the non-parabolic version of the Deligne-Hitchin moduli space and how it is proved from the Hodge moduli space. A Torelli theorem is stated for both moduli spaces.

Finally, the last chapter presents the main original result of this master thesis. Adapting the techniques used in [BGHL09] and [BGH13] to the parabolic case, a new alternative proof of the Torelli theorem for the moduli space of semistable $\mathrm{SL}(r,\mathbb{C})$-Higgs bundles is given. The parabolic version of the Hodge moduli space for $\mathrm{SL}(r,\mathbb{C})$-bundles is described and a new Torelli theorem for this moduli space is proved as well. Finally, using a parabolic version of the Riemann-Hilbert correspondence, the parabolic Deligne-Hitchin moduli space is defined and a new Torelli theorem for the parabolic Deligne-Hitchin moduli space is stated.

Therefore, two completely new results are given, the Torelli theorem for the Hodge moduli space of semistable parabolic $\mathrm{SL}(r,\mathbb{C})$-$\lambda$-connections and the Torelli theorem for the parabolic Deligne-Hitchin moduli space. Moreover, an alternative proof of the Torelli theorem for the moduli space of semistable parabolic Higgs bundles is derived.

Chapter 1

Previous concepts

In this chapter we will introduce some of the basic concepts, theorems and notation that will be used along this work. The main references are [Har10], [Gro55] and [iB]. Additional information can be found in [Ram06] and [HJBS08].

1.1 Fibre spaces

Definition 1.1.1. *A fibre space over a topological space X is a triple (X, E, p) of the space X, a topological space E and a continuous map $p : E \to X$.*

For a given $x \in X$, the space $p^{-1}(x)$ is called the fibre over x and will be denoted by E_x. Given two fibre spaces (X, E, p) and (X', E', p'), a homomorphism of the first into the second is a pair of continuous maps $f : X \to X'$ and $g : E \to E'$ such that the following diagram commutes

$$\begin{tikzcd} E \arrow[r, "g"] \arrow[d, "p"'] & E' \arrow[d, "p'"] \\ X \arrow[r, "f"] & X' \end{tikzcd} \tag{1.1.1}$$

If the map $f : X \to X'$ is fixed, g will be called an f-homomorphism of E into E'. The pair (f, g) is an isomorphism of fibre spaces if f, g are invertible and the pair (f^{-1}, g^{-1}) is a homomorphism. Equivalently, (f, g) is an isomorphism if both f and g are homeomorphisms onto X' and E' respectively.We will take the usual notation $(X, E, p) \cong (X', E', p')$ to denote that there exist an isomorphism between (X, E, p) and (X', E', p').

If (X, E, p) is a fibre space over X and $f : X' \to X$ is a continuous map, we define the inverse image or pullback of the fibre space E by f to be the subspace of $X' \times E$ of points (x', y) such that $f(x') = p(y)$. We will denote this subspace by f^*E

The inverse image of a fibre space is clearly a fibre space taking the map $p' : f^*E \to X$ given by $p'(x', y) = x'$. Moreover, the map $g : f^*E \to E$ defined by $g(x', y) = y$ is by construction an f-homomorphism, inducing for each $x' \in X'$ a homeomorphism of the fibre of f^*E over x' onto the fibre of E over $f(x')$.

If Y is a subspace of X and $i : Y \hookrightarrow X$ is the inclusion of Y into X, the pullback of X by i is called the fibre space induced by X on Y or the restriction of E to Y and it's denoted by $E|_Y$. We have a canonical homeomorphism between $E|_Y$ and the subspace $p^{-1}(Y)$.

Definition 1.1.2. *Two spaces E, E' over X are said to be locally isomorphic if each point $x \in X$ has a neighborhood U such that $E|_U$ and $E'|_U$ are isomorphic as fibre spaces.*

One of the most important yet trivial example of fibre space is the following. If X, F are two topological spaces and $\pi : X \times F \to X$ is the canonical projection, the triple $(X, X \times F, \pi)$ is a fibre space called the trivial fibre space over X with fibre F. A fibre space (X, E, p) will be called trivial if there exist a topological space F such that $(X, E, p) \cong (X, X \times F, \pi)$.

We will now focus on fibre spaces that have a fixed fibre over each point, that is, fibre spaces (X, E, p) such that for every $x \in X$, $p^{-1}(x)$ is isomorphic to a certain fixed topological space F.

Definition 1.1.3. *If F is a given topological space, a fibre space (X, E, p) is said locally trivial with fibre F if it is locally isomorphic to the trivial space $(X, X \times F, \pi)$.*

For a fixed base space X and fibre F, there may exist multiple nonisomorphic locally trivial vector bundles. For example, taking $X = S^1$ and $F = (0, 1)$, the Mbius strip $[0, 1] \times (0, 1)/\sim$, where $(0, x) \sim (1, 1 - x)$ and the cilinder $[0, 1] \times (0, 1)/\sim'$, where $(0, x) \sim' (1, x)$ are both locally trivial vector bundles but are not isomorphic, as the cilinder is trivial $S^1 \times (0, 1)$ and the Mobius strip is not.

As we have defined fibre spaces and its main properties in a topological way, we may restrict our definitions to narrower categories. We will say that a fibre space is smooth whenever it is a fibre space in the category of differentiable manifolds. In an analogous way, we can talk about algebraic fibre spaces or holomorphic fibre spaces.

Another way of describing fibre spaces is via transition functions. Let X be a topological space, $(U_i)_{i \in I}$ a covering of X. For each index $i \in I$ let E_i be a fibre space over U_i and for every pair $i, j \in I$ such that $U_{ij} = U_i \cap U_j \neq \emptyset$, let f_{ij} be an isomorphism of $E_j|_{U_{ij}}$ onto $E_i|_{U_{ij}}$. Suppose that for every i, j, k such that $U_{ijk} = U_i \cap U_j \cap U_k \neq \emptyset$ the following cocycle condition holds

$$f_{ik}|_{U_{ijk}} = f_{ij}|_{U_{ijk}} \circ f_{jk}|_{U_{ijk}} \tag{1.1.2}$$

Let $\mathcal{E}$ be the topological sum of the spaces E_i. Let us consider the relation given by $y_i \sim y_j$ for some $y_i \in E_i|_{U_{ij}}$ and $y_j \in E_j|_{U_{ij}}$ if and only if $y_i = f_{ij}(y_j)$. The cocycle condition (1.1.2) is necessary and sufficient for this relation to be an equivalence relation. Let E be the quotient $\mathcal{E}/\sim$. The projections $p_i : E_i \to U_i$ define a continuous map $(p_i) : \mathcal{E} \to X$. As f_{ij} are isomorphisms of fibre spaces, the equivalence relation is compatible with the projections p_i. Therefore, the projections p_i induce a continuous map $p : E \to X$.

Definition 1.1.4. *The fibre space (X, E, p) defined above is called the fibre space defined by the transition functions (f_{ij}) between the fibre spaces E_i.*

As we have taken the equivalence relation to identify common fibers of the fibre spaces E_i, the inclusion $E_i \hookrightarrow \mathcal{E}$ induces a map $\varphi_i : E_i \to E$ which is a bijective homomorphism of E_i onto $E|_{U_i}$. Moreover, the following proposition holds.

Proposition 1.1.5. *For every $i \in I$, φ_i is an isomorphism of E_i onto $E|_{U_i}$.*

Proof. Let $\pi : \mathcal{E} \to E$ is the canonical projection. Then the following diagram commutes

$$\begin{tikzcd}
E_i \arrow[rr, hook] \arrow[dr, "p_i"] \arrow[ddr, "\varphi_i"'] & & \mathcal{E} \arrow[dl, "(p_i)"'] \arrow[ddl, "\pi"] \\
& X & \\
& E \arrow[u, "p"] &
\end{tikzcd} \tag{1.1.3}$$

As we know that φ_i is a bijective homomorphism of fibre spaces, in order to prove that φ_i is a fibre space isomorphism it is enough to prove that it is open.

Let $V \subseteq E_i$ be an open set. By definition of the quotient topology, the topology of $E = \mathcal{E}/\sim$ is the finest for which the map π is continuous. Then $\varphi(V)$ is open if an only if $\pi^{-1}(\varphi(V))$ is open in $\mathcal{E}$. Let $I_i = \{j : U_i \cap U_j \neq \emptyset\}$. By construction of the equivalence relation, the class of a point $y \in E_i$ consists on the images of that point under the maps f_{ji} for every $j \in I_i$ such that $p_i(y) \in U_j$. Thus,

$$\pi^{-1}(\varphi(V)) = \bigcup_{j \in I_i} f_{ji}(V \cap p_i^{-1}(U_j \cap U_i))$$

For every $i \in I$, $p_i : E_i \to U_i$ is continuous so $V \cap p_i^{-1}(U_j \cap U_i)$ is open. As f_{ji} is a homeomorphism for every $j \in I_i$, $f_{ji}(V \cap p_i^{-1}(U_j \cap U_i))$ is open for every $j \in I_i$. Finally, the union is in fact a topological sum in $\mathcal{E}$, so $\pi^{-1}(\varphi(V))$ is open.

The construction of φ_i and the equivalence relation $\sim$ implies that for every i, j such that $U_i \cap U_j \neq \emptyset$

$$f_{ij} = \varphi_i^{-1}|_{U_i \cap U_j} \circ \varphi_j|_{U_i \cap U_j} \tag{1.1.4}$$

□

On the other side, let (X, E, p) be a fibre space. Let $(U_i)|_{i \in I}$ be a covering of X and for every $i \in I$ let E_i be a fibre space over U_i such that there exist an isomorphism $\varphi_i : E_i \to E|_{U_i}$. Then for each $i, j \in I$ such that $U_i \cap U_j \neq \emptyset$, equation (1.1.4) defines transition functions (f_{ij}) that clearly satisfy the cocycle condition (1.1.2).

It can be proved [Gro55] that the fibre space defined by the transition functions (f_{ij}) and the fibre spaces E_i is isomorphic to (X, E, p).

In the case of locally trivial fibre spaces the situation is much simpler. A locally trivial fibre space over X with fibre F is completely determined by an open covering of X, $(U_i)_{i \in I}$ and transition functions (f_{ij}) that satisfy condition (1.1.2).

1.2 Vector bundles

Let X be a scheme over an algebraically closed field k. Let $\mathbb{A}^r$ denote the affine space of dimension r over k, i.e. $\mathbb{A}^r = \operatorname{Spec} k[x_1, \ldots, x_r]$.

Definition 1.2.1. *A vector bundle of rank r over X is an algebraic locally trivial space (X, E, p) with fibre $\mathbb{A}^r$ such that there exist an open affine covering $\{U_i\}_{i \in I}$ of X and isomorphisms*

$$\varphi_i : p^{-1}(U_i) \to U_i \times \mathbb{A}^r$$

such that for every $i, j \in I$ such that $U_i \cap U_j \neq \emptyset$ there exist $\varphi_{ij} : U_{ij} \to \mathrm{GL}(r, k)$ such that

$$\varphi_j \circ \varphi_i^{-1}|_{U_i \cap U_j} = (\mathrm{Id}, \varphi_{ij})$$

A morphism between two vector bundles is a fibrewise linear fibre space morphism of locally constant rank.

Taking into account the definition of the coordinate transformations given in the last section, a vector bundle over X is simply a locally trivial fibre bundle over X with fibre $\mathbb{A}^r$ and linear coordinate transformations.

A vector bundle of rank 1 is called a line bundle. The usual operations between vector spaces such as direct sum, tensor product, wedge product or taking the dual can be extended to vector bundles fibrewise. A specially relevant case is the determinant of a vector bundle.

Definition 1.2.2. *Let (X, E, p) be a vector bundle of rank r. Let $(U_i)_{i \in I}$ be a covering on which E is locally trivial and let $\varphi_{ij} : U_{ij} \to \mathrm{GL}(r, k)$ be the corresponding transition functions. The determinant of (X, E, p) is the vector bundle $\det(E) := \bigwedge^r E$ over X with fiber $\bigwedge^r E_x$ over each point $x \in X$ and transition functions $\det(\varphi_{ij}) : U_{ij} \to \mathrm{GL}(1, k)$.*

Alternatively, we can describe vector bundles over a scheme X in terms of sheaf theory. Let $\mathcal{O}_X$ be the structure sheaf of X.

Definition 1.2.3. *A locally free sheaf of rank r on X is a sheaf of modules $\mathcal{E}$ such that there exist an open covering $(U_i)_{i \in I}$ of X and $\mathcal{E}(U_i) \cong (\mathcal{O}_X(U_i))^r$ for every $i \in I$.*

Theorem 1.2.4. *There is a natural one to one correspondence between vector bundles and locally free sheaves on X.*

Proof. Let (X, E, p) be a vector bundle over X. We build the associated sheaf $\mathcal{E}$ as follows. For every open set $U \subseteq X$ we define $\mathcal{E}(U)$ to be the $\mathcal{O}_X(U)$-module of sections of $E|_U$. Two sections can be added by adding their evaluations on each fiber in the corresponding vector space. A section and a function of $\mathcal{O}_X(U)$ can be multiplied pointwise on each fibre. We need to show that $\mathcal{E}$ is locally free.

Let $(U_i)_{i \in I}$ be a covering of X such that $E|_{U_i}$ is trivial. Let $\varphi_i : E|_{U_i} \to U_i \times \mathbb{A}^r$ the local trivializations. Let us consider the canonical local coordinate sections over

U, $x_j : U_i \to E|_{U_i}$ given by $x_j(p) = \varphi_i^{-1}(p, (0, \ldots, \overset{j}{1}, \ldots, 0))$. Clearly every section s over U_i can be uniquely written as $s = f_1 x_1 + \ldots + f_r x_r$ for some functions $f_1, \ldots, f_r \in \mathcal{O}_X(U_i)$. Hence the morphism from $\mathcal{E}(U_i)$ to $(\mathcal{O}_X(U_i))^r$ sending s to $(f_1, \ldots, f_r)$ gives the required isomorphism.

Conversely, let $\mathcal{E}$ be a locally free sheaf. For every closed point $x \in X$, let $\mathcal{M}_x$ be the maximal ideal of the local ring $\mathcal{O}_{X,x}$. We define the fibre space

$$E = \{(x, t) : x \in X, t \in \mathcal{E}_x / \mathcal{M}_x \mathcal{E}_x\}$$

with the map $p : E \to X$ given by the projection to the first component. As k is algebraically closed, $\mathcal{O}_{X,x}/\mathcal{M}_x \cong k$. Therefore, if U is an open set on which $\mathcal{E}(U) \cong \mathcal{O}_X(U)^r$, then for every $x \in U$, $\mathcal{E}_x/\mathcal{M}_x\mathcal{E}_x \cong \mathbb{A}^r$. We need to show that it is locally trivial and that the transition functions are linear.

Let (U_i) be an affine covering of X over which $\mathcal{E}$ is locally free. Over each U_i, $E|_{U_i}$ coincides with the étale space of $\mathcal{E}|_{U_i} \cong \mathcal{O}_X(U_i)^r$, so E is free over each U_i.

Over $U_i \cap U_j$, the isomorphisms $\mathcal{E}(U_i) \cong \mathcal{O}_X(U_i)^r$ and $\mathcal{E}(U_j) \cong \mathcal{O}_X(U_j)^r$ induce an isomorphism of $\mathcal{O}_X(U_i \cap U_j)$-modules $A_{ij} : \mathcal{O}_X(U_i \cap U_j)^r \to \mathcal{O}_X(U_i \cap U_j)^r$. A_{ij} has the form of a matrix of sections of $\mathcal{O}_X(U_i \cap U_j)$. Then we can define the transition functions on the intersection $U_i \cap U_j$ as $(\mathrm{Id}, \varphi_{ij})$, where $\varphi_{ij}(x)$ is the localization of A_{ij} at x reduced moduli $\mathcal{M}_x$. As $\mathcal{O}_{X,x}/\mathcal{M}_x \cong k$, we get $\varphi_{ij} : U_{ij} \to \mathrm{GL}(k, r)$. The cocycle condition (1.1.2) is clearly satisfied by construction.

We define the associated algebraic vector bundle as the locally trivial vector bundle trivial over (U_i) with transition functions φ_{ij}. The previous observations imply that its underlying topological space is homeomorphic to E. □

If X is a smooth projective curve we can characterize locally free sheafs in terms of its torsion.

Definition 1.2.5. *A sheaf $\mathcal{E}$ of $\mathcal{A}$-modules is a torsion sheaf if for each open set $U \subset X$, $\mathcal{E}(U)$ is a torsion ring.*

Definition 1.2.6. *A sheaf $\mathcal{E}$ of $\mathcal{A}$-modules is torsion-free if for each open set $U \subset X$, $\mathcal{E}(U)$ has no torsion elements.*

The existence of torsion elements is clearly a local property and thus can be stated either in local terms, through a covering or as a property of the sheaf stalks.

Proposition 1.2.7. *Let $\mathcal{E}$ be a sheaf of $\mathcal{A}$-modules over an integral scheme X. The following statements are equivlaent.*

a) $\mathcal{E}$ is a locally free sheaf.

b) There exist a covering $(U_i)_{i \in I}$ of X such that $\mathcal{E}(U_i)$ has no torsion elements for every $i \in I$.

c) For every $x \in X$ and $f \in \mathcal{O}_{X,f} \backslash \{0\}$ multiplication by f is an injective isomorphism $\mathcal{E}_x \to \mathcal{E}_x$.

Proof. It is clear that (a) implies (b). Let us see that (a) implies (c). Suppose that there exist $x \in X$ and $f \in \mathcal{O}_{X,f} \setminus \{0\}$ such that the product by f is not an isomorphism. Then there is an element $s \in \mathcal{E}_x$ such that $fs = 0$. Let U be a sufficiently small open neighborhood of x, so that there exist representatives (U, F) and (U, S) of f and s respectively such that (U, FS) belongs to the zero class in $\mathcal{E}_x$ and thus, FS is constantly zero on U. Therefore, S would be a torsion element of $\mathcal{E}(U)$.

To prove that (b) implies (a), suppose that there exist an open set V such that $\mathcal{E}(V)$ has a torsion element s. Let $f \in \mathcal{O}_X(V)$ be a function such that $f \neq 0$ and $fs = 0$ on V. Let $V_i := V \cap U_i$. Let V_i be one of those open sets such that s and f are nonzero over V_i.Restricting s and f to V_i we obtain that s is a torsion element of $\mathcal{E}(V_i)$. As V is affine, V_i is an open subset of $\operatorname{Spec} A$. Therefore, V_i is of the form $\operatorname{Spec} A_p$. Thus, s is of the form $\frac{s'}{p^a}$ for some $s' \in \mathcal{E}(V_i)$, and f is of the form $\frac{f'}{p^b}$ for some $f' \in \mathcal{O}_X(V_i)$. Then s' is a torsion element of $\mathcal{E}(V_i)$.

Let us finally prove that (c) implies (a). Suppose that we have an open set U and $s \in \mathcal{E}(U)$, $f \in \mathcal{O}_X(U)$ such that $f \neq 0$ and $fs = 0$. As $s \neq 0$ and $f \neq 0$, there exist a point $x \in X$ such that the classes $[(U, s)] \in \mathcal{E}_x$ and $[(U, f)] \in \mathcal{O}_{X,x}$ are nonzero. Let us consider the morphism $\mathcal{E}_x \to \mathcal{E}_x$ given by the product by $[(U, f)]$. As $fs = 0$ on U, it maps $[(U, s)]$ to zero, so it can't be an isomorphism. □

Condition (b) implies that it is enough to find a locally torsion-free covering of a sheaf in order for the sheaf to be torsion-free. As a corollary we obtain the following proposition.

Corollary 1.2.8. *Evey locally free sheaf is torsion-free.*

Proof. Let $\mathcal{E}$ be a locally free sheaf of rank r. Let $(U_i)_{i \in I}$ be trivialization of $\mathcal{E}$ such that $\mathcal{E}(U_i) \cong \mathcal{O}_X(U)^r$. As $\mathcal{E}(U_i)$ is a free $\mathcal{O}_X(U_i)$-module, all its sections $s \in \mathcal{E}(U_i)$ can be written as a $\mathcal{O}_X(U_i)$- linear combination of the canonical generators. As the canonical generators vanish nowhere, the support of all the nonzero sections is open and thus there can't exist a nonzero function $f \in \mathcal{O}_X(U_i)$ such that $fs = 0$. Therefore, none of the sections on U_i are torsion elements and $\mathcal{E}$ is torsion free on U_i for every $i \in I$. By condition (b) of the previous proposition, $\mathcal{E}$ is torsion-free. □

The reciprocal is generally not true. We must impose some extra conditions to the scheme X.

Proposition 1.2.9. *If X is a smooth curve then every torsion-free coherent sheaf is locally free.*

Proof. Let p be any point in X. If X is a smooth curve, there exist an open set U isomorphic to the spectrum of a ring R such that R_p is principal ideal domain. If $\mathcal{E}$ is a coherent sheaf on X, then $\mathcal{E}_p$ is a finitely generated R_p-module. By the structure theorem for finitely generated modules over a principal ideal domain [AM69], there is a unique decreasing sequence of proper ideals $(d_1) \supseteq (d_2) \supseteq \cdots \supseteq (d_n)$ such that $\mathcal{E}_p \cong \bigoplus_{j=1}^n R_p/(d_j)$.

If (d_i) is nonzero for some i, then $1 \in R_p/(d_j)$ is a torsion element in $R_p/(d_j)$ and thus, $\mathcal{E}_p$ would contain a torsion element. As $\mathcal{E}$ is torsion free, then $d_i = 0$ for all $i = 1, \ldots, n$, so $\mathcal{E}_p \cong (R_p)^n$. Let $s_1, \ldots, s_n$ be generators of $\mathcal{E}_p$. There exist an open set $V \subseteq U$ such that there is a representant $\tilde{s_i} \in \mathcal{E}(V)$ for every generator $s_i \in \mathcal{E}_p$. For every q in an open subset of $V' \subseteq V$ that contains p, $\{(\tilde{s_i})_q\}$ generate $\mathcal{E}_q$. Therefore, $\{\tilde{s_i}|_{V'}\}$ generate $\mathcal{E}(V')$ and $\mathcal{E}(V')$ is free. As this holds for every point in X, $\mathcal{E}$ is locally free. □

1.3 Line bundles and divisors

In this section we will recall some of the basic properties of divisors over a curve and its relation to line bundles. [Har10, II.6] gives a complete overview of the subject, so we will only introduce some of the main definitions and theorems in order to fix the notation. Let X be a noetherian integral separated scheme which is regular in codimension one.

Definition 1.3.1. *A prime divisor on X is a closed integral subscheme Y of codimension one. A Weil divisor is an element of the free abelina group generated by the prime divisor.*

We will call Div the group of Weil divisors over X. A Weil divisor $D \in \mathrm{Div}(X)$ can be expressed as $D = \sum_{i=1}^{k} n_i Y_i$ for some closed integral subschemes Y_i of codimension one. If X has dimension one then the only closed integral subschemes of codimension one are the closed points of X, so $\mathrm{Div}(X)$ is the free group generated by the closed points of X and a Weil divisor on X is simply a formal sum of points.

If X has dimension one, the degree if a divisor $D = \sum_{i=1}^{k} n_i Y_i$ is $\deg(D) := \sum_{i=1}^{k} n_i$. If $f : X \to k^*$ is a rational function, the divisor associated to f is

$$(f) := \sum_{x \in X} \mathrm{ord}_x(f) x$$

where the sum is taken over the prime divisors on X. Divisors D such that $D = (f)$ for some rational function f are called principal divisors. Two divisors D and D' are said to be linearly equivalent, written $D \sim D'$ whenever $D - D'$ is a principal divisor.

For each open subset $U = \operatorname{Spec} A$, let S be the set of elements of A which are not zero divisors and let $K(U)$ be the localization of A by the multiplicative system S. We call $K(U)$ the total quotient ring of A. For each open set U, let $S(U)$ denote the set of elements of $\mathcal{O}_X(U)$ which are not zero divisors in each local ring $\mathcal{O}_{X,x}$ for $x \in U$. Then the rings $S(U)^{-1}\mathcal{O}_X(U)$ form a presheaf, whose associated sheaf of rings $\mathcal{K}$ will be called the sheaf of total quotient rings of $\mathcal{O}_X$.

Definition 1.3.2. *A Cartier divisor on a scheme X is an open cover $(U_i)_{i \in I}$ of X together with an element $f_i \in \mathcal{K}^*(U_i)$ for every $i \in I$, such that for every i, j, $f_i/f_j \in \mathcal{O}_X^*(U_i \cap U_j)$.*

A Cartier divisor is principal if it is in the image of the natural map $\mathcal{K}^*(X) \to (\mathcal{K}^*/\mathcal{O}_X^*)(X)$. Two Cartier divisors are linearly equivalent if their difference if principal.

Proposition 1.3.3. *Let X be a smooth curve. Then there is a one to one correspondence between Weil divisors and Cartier divisors on X.*

Proof. Let $\{(U_i, f_i)\}$ be a Cartier divisor. We define the associated Weil divisor as follows. For every closed point $x \in X$, we take the coefficient of x to be $\mathrm{ord}_x(f_i)$, where i is any index such that $x \in U_i$. If i, j are such that $x \in U_i \cap U_j$, then f_i/f_j is invertible on $U_i \cap U_j$, so $\mathrm{ord}_x(f_i/f_j) = 0$ and $\mathrm{ord}_x(f_i) = \mathrm{ord}_x(f_j)$. Thus, we obtain a well defined Weil divisor $D = \sum_{x \in X} \mathrm{ord}_x(f_i)$ on X.

Conversely, let D be a Weil divisor on X. For every closed point $x \in X$, D induces a Weil divisor D_x on the local scheme $\operatorname{Spec} \mathcal{O}_{X,x}$. Since $\mathcal{O}_{X,x}$ is a principal ideal domain, there exist $f_x \in K$ such that $D_x = (f_x)$. The principal divisor (f_x) on X has the same restriction to $\operatorname{Spec} \mathcal{O}_{X,x}$ as D. Therefore, they differ only at points different from x. There are only finitely many of these which have a non-zero coefficient in D or (f_x), so there is an open neighborhood U_x of x such that D and (f_x) have the same restriction to U_x. Covering X with such open sets $(U_x)_{x \in X}$, the functions f_x give a Cartier divisor on X.

□

Definition 1.3.4. *Let D be a Cartier divisor on a scheme X, represented by $\{(U_i, f_i)\}$. We define the sheaf associated to D, $L(D)$, as the subsheaf of total quotient ring K determined by taking $L(D)$ to be the sub-$\mathcal{O}_X$-module of $\mathcal{K}$ generated by f_i^{-1} on U_i.*

The sheaf $L(D)$ is well defined, as f_i/f_j is invertible on $U_i \cap U_j$, so f_i^{-1} and f_j^{-1} generate the same sub-$\mathcal{O}_X(U_i \cap U_j)$-module. As it is locally generated by f_i^{-1} on U_i, $L(D)$ results to be a rank 1 locally free sheaf, trivial over (U_i).

Corollary 1.3.5. *For every line bundle L on a scheme X there exist a divisor D such that $L = L(D)$.*

Proof. Let (U_i) be a covering for X on which L is locally trivial with isomorphisms $\varphi_i : L|_{U_i} \to U_i \times \mathbb{A}^1$. Let φ_{ij} be the corresponding transition functions. Fix an open set U_i and consider the constant section $s : U_i \to k$ given by $s(x) = \varphi_i^{-1}(x, 1)$ for every $x \in U$. For every j, let us define $s(x) = \varphi_j^{-1}(x, \frac{1}{\varphi_{ji}(1)})$. s is well defined because of the cocycle condition (1.1.2) for φ_{ij}. As all the transition functions are algebraic, s is an algebraic section and defines a Cartier divisor. By construction, the associated line bundle to that Cartier divisor is exactly L. □

It can be proved [Har10, Proposition II.6.13] and [Har10, Corollary II.6.16] that the correspondence between Cartier divisors and line bundles over a curve previously described has the following properties

Proposition 1.3.6. *Let X be a curve. Then*

a) The map $D \to L(D)$ is a one to one correspondence between Cartier divisors on X and line bundles on X.

b) $L(D_1 - D_2) \cong L(D_1) \otimes L(D_2)^{-1}$.

c) $D_1 \sim D_2$ if and only if $L(D_1) \cong L(D_2)$.

We can use this correspondence to build a numeric invariant on line bundles from the degree of divisors.

Definition 1.3.7. *The degree of a line bundle L is the degree of any divisor D such that $L(D) = L$.*

The degree of a line bundle is well defined. It is invariant under line bundle isomorphism, because two divisors that induce isomorphic line bundles are linearly equivalent, i.e., they differ in the divisor of a rational function. The degree of the divisors doesn't change because the degree of the divisor of a rational function is always zero.

The degree defined this way is an additive function in the following sense, if A, B and C are line bundles such that

$$B \cong A \otimes C$$

we have $\deg(A) + \deg(C) = \deg(B)$. We can extend the degree definition for a general vector bundle through the determinant bundle.

Definition 1.3.8. *The degree of a vector bundle E is the degree of* $\det(E)$*,*

$$\deg(E) := \deg(\det(E))$$

The following proposition proves that both the degree and the rank of a vecto bundle are additive functions.

Proposition 1.3.9. *Let A, B, C be vector bundles such that the sequence*

$$0 \to A \to B \to C \to 0$$

is exact. Then

a) $\det(B) \cong \det(A) \otimes \det(C)$

b) $\deg(A) + \deg(C) = \deg(B)$.

c) $\mathrm{rk}(A) + \mathrm{rk}(C) = \mathrm{rk}(B)$.

Proof. Let $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{C}$ be the corresponding locally free sheaves. As the rank is constant, in order to prove (c) it is enough to consider an affine open set $U \subset X$ on which $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{C}$ are free. Then $0 \to \mathcal{A}(U) \to \mathcal{B}(U) \to \mathcal{C}(U) \to 0$. As U is affine, $\mathcal{A}(U)$, $\mathcal{B}(U)$ and $\mathcal{C}(U)$ are free $\mathcal{O}_X(U)$-modules and the additivity for the rank is directly derived from the additivity of the dimension of the modules. As a consequence of (a), we get that

$$\deg(B) = \deg(\det(A) \otimes \det(C)) = \deg(A) + \deg(C)$$

Let us prove (a). Let (U_i) be a covering for X over which A, B and C are trivial. Restricting the exact sequence over U_i we get the following short exact sequence of trivial vector bundles

$$0 \to A|_{U_i} \to B|_{U_i} \to C|_{U_i} \to 0$$

so that $B|_{U_i} \cong A|_{U_i} \otimes C|_{U_i}$. Let $U_{ij} = U_i \cap U_j \neq \emptyset$. Let a_i, b_i, c_i the corresponding trivializations for A, B and C over U_i and U_j. Then we have the following commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & U_{ij} \times \mathbb{A}^a \arrow[r] \arrow[d, "a_i^{-1}"] & U_{ij} \times \mathbb{A}^a \oplus \mathbb{A}^c \arrow[r] \arrow[d, "b_i^{-1}"] & U_{ij} \times \mathbb{A}^c \arrow[r] \arrow[d, "c_i^{-1}"] & 0 \\
0 \arrow[r] & A|_{U_{ij}} \arrow[r] \arrow[d, "a_j"] & B|_{U_{ij}} \arrow[r] \arrow[d, "b_j"] & C|_{U_{ij}} \arrow[r] \arrow[d, "c_j"] & 0 \\
0 \arrow[r] & U_{ij} \times \mathbb{A}^a \arrow[r] & U_{ij} \times \mathbb{A}^a \oplus \mathbb{A}^c \arrow[r] & U_{ij} \times \mathbb{A}^c \arrow[r] & 0
\end{tikzcd} \tag{1.3.1}$$

Therefore, we get that $b_j \circ b_i^{-1}$ has the form

$$b_j \circ b_i^{-1} = \begin{pmatrix} (a_j \circ a_i^{-1}) & d_{ij} \\ 0 & (c_j \circ c_i^{-1}) \end{pmatrix}$$

where $d_{ij} : U_{ij} \times \mathbb{A}^c \to U_{ij} \times \mathbb{A}^a$ is fiberwise linear. Taking the determinants we have that $\det(b_j \circ b_i^{-1}) = \det(a_j \circ a_i^{-1}) \det(c_j \circ c_i^{-1})$. Therefore, for the trivialization over (U_i), the transition functions for $\det(B)$ and $\det(A) \otimes \det(C)$ are the same. As they are locally isomorphic over (U_i), we get that $\det(B) \cong \det(A) \otimes \det(C)$. □

The degree of a vector bundle defined from its determinant gives a lot of information about its topology. In fact, the Riemann-Roch theorem relates the degree of a vector bundle to its Euler-Poincaré characteristic, defined as follows

Definition 1.3.10. *Let $\mathcal{F}$ be a sheaf on a projective scheme X. Let $h^i(X, \mathcal{F})$ denote the dimension of the sheaf cohomology group $H^i(X, \mathcal{F})$. We define the Euler-Poincaré characteristic of $\mathcal{F}$ as*

$$\chi(\mathcal{F}) := \sum_i (-1)^i h^i(X, \mathcal{F})$$

If X is a scheme of dimension one, $H^i(X, \mathcal{F})$ is zero for every $i > 1$, so the Euler-Poincaré characteristic reduces to $\chi(\mathcal{F}) = h^0(X, \mathcal{F}) - h^1(X, \mathcal{F})$. As the sheaf cohomology is exact, if $0 \to A \to B \to C \to 0$ is a short exact sequence of sheafs, then there exist a long exact sequence of cohomology groups

$$0 \to H^0(X, A) \to H^0(X, B) \to H^0(X, C) \to H^1(X, A) \to H^1(X, B) \to \cdots$$

As the dimension is additive, we get that

$$\chi(A) - \chi(B) + \chi(C) = \sum_i (-1)^i (h^i(X, A) - h^i(X, B) + h^i(X, C)) = 0$$

Thus, the Euler-Poincaré characteristic is additive.

Theorem 1.3.11 (Riemann-Roch)**.** *Let E be a vector bundle of rank r over a curve X of genus g. Then*

$$\deg(E) = \chi(E) - r(1 - g)$$

This theorem allows us to define alternatively the degree of a vector bundles in terms of its Euler-Poincaré characteristic and rank. This definition can be extended to torsion sheaves. Let $\mathcal{E}$ be a torsion sheaf over a smooth curve X. By the structure theorem of finitely generated modules over a principal ideal domain, over an affine open subset $U \subseteq X$, $\mathcal{E}(U)$ has the form

$$\mathcal{E}(U) \cong \bigoplus_i R/(d_i)$$

where d_i form a sequence of decreasing proper ideals $(d_1) \supseteq (d_2) \supseteq \cdots \supseteq (d_n)$. Let us define the rank of $\mathcal{E}$ to be the number of ideals (d_i) that are zero. If $\mathcal{E}$ is a torsion sheaf, all the elements of $\mathcal{E}(U)$ must be torsion elements, so $(d_i) \neq 0$ for all i. Thus, it is natural to define $\mathcal{E}$ to have zero rank.

We can now extend the definition of the degree for torsion sheaves, treating them as "rank 0" sheaves

Definition 1.3.12. *Let $\mathcal{E}$ be a torsion sheaf over a curve X. We define the degree of $\mathcal{E}$ as*

$$\deg(\mathcal{E}) := \chi(\mathcal{E})$$

As X is a curve, then $\chi(\mathcal{E}) = h^0(X, \mathcal{E}) - h^1(X, \mathcal{E})$. As $\mathcal{E}$ is a torsion sheaf, $\mathcal{E}(X)$ has torsion elements, so $\mathcal{E}$ must be supported on a closed subscheme of X. Therefore, $H^1(X, \mathcal{E}) = 0$ and we get the alternate definition

$$\deg(\mathcal{E}) = h^0(X, \mathcal{E}) \tag{1.3.2}$$

As the Euler-Poincaré characteristic and the rank are both additive functions, the generalized degree is clearly additive.

1.4 Subbundles and subsheaves

Once we have stated the main properties of vector bundles, we will study some properties about vector bundle morphisms and sheaf morphisms.

Whereas we have proved that vector bundles and locally free sheaves are essentially the same objects, the morphisms between vector bundles are not exactly the same as sheaf morphisms. By construction of the underlying locally free sheaf of a vector bundle, it is clear that a vector bundle morphism is a locally free sheaf morphism, but the reciprocal is not true in general. The key is the local rank of the morphism. A morphism of vector bundles has always a locally constant rank and thus, it has a constant rank for connected schemes. On the other hand, a morphism of sheaves does not have a fixed rank a priori.

As an example, let X be a smooth curve, i.e. $X = \mathbb{A}^1$. Let $x \in X$ be a close point, i.e., $x = (0)$. We denote by $\mathcal{O}_X(-x)$ the line bundle of local morphisms $f : U \to k$ such that $f(x) = 0$. We will later on describe this line bundle in terms of the corresponding divisor on the curve. Let us consider the inclusion morphism $\mathcal{O}_X(-x) \hookrightarrow \mathcal{O}_X$.

This inclusion can't be a vector bundle morphism, as $\mathcal{O}_X(-x)$ and $\mathcal{O}_X$ have both rank one. If the morphism was a vector bundle morphism, its image would be a vector bundle that would be a subbundle of the rank one bundle $\mathcal{O}_X$. As $\mathcal{O}_X(-x) \neq \mathcal{O}_X$, the only possible subbundle is the trivial one, but $\mathcal{O}_X(-x)$ is not trivial.

On the other hand, the inclusion is clearly a sheaf morphism. The obstruction for it to be a vector bundle morphism lies at the point x. Seen as line bundles, the morphism at the fibre over x is exactly the zero morphism, so it has rank zero, but the morphism at any other fibre is the identity morphism, so it has rank one.

As the category of vector bundles has been defined completely, there exist a clear notion of subobject in the category, i.e., a subbundle of a vector bundle is a monomorphism. As the category of vector bundles is a subcategory of the category of locally free sheaves with exactly the same objects, it is natural to study the relations between subbundles and subsheafs of a fixed vector bundle.

The following lemma provides an essential step in order to understand this relation.

Lemma 1.4.1. *A subsheaf of a locally free sheaf over a smooth curve is locally free.*

Proof. On a smooth curve, being locally free is equivalent to being torsion free. Let $\mathcal{F}$ be a subsheaf of $\mathcal{E}$. Let $i : \mathcal{F} \to \mathcal{E}$ be the inclusion morphism. Suppose that $\mathcal{F}$ is not locally free. Then it is not torsion-free, so there is an open set $U \subseteq X$ such that $\mathcal{F}(U)$ has a torsion element s. Thus there exist $f \in \mathcal{O}_X(U)\backslash\{0\}$ such that $fs = 0$ over U. As i is a morphism of sheaves of $\mathcal{O}_X$-modules, we get that

$$fi(U)(s) = i(U)(fs) = i(U)(0) = 0$$

As $i(U)$ is injective, $i(U)(s) \neq 0$, so $i(U)(s)$ is a torsion element in $\mathcal{E}(U)$. □

Corollary 1.4.2. *Let E and F be vector bundles. F is a subbundle of E if and only if it is a subsheaf such that the inclusion morphism $i : F \hookrightarrow E$ has locally constant rank.*

Proof. The definition of vector bundle morphism implies that a subbundle is a subsheaf such that the inclusion has locally constant rank. Let F be a subsheaf of E with locally constant rank. The previous lemma shows that F is a vector bundle. The morphism $i : F \to E$ is a sheaf monomorphism of locally constant rank, so it is a vector bundle monomorphism. Thus, F is subbundle of E. □

Therefore, we can give the following alternative definition of subbundle.

Definition 1.4.3. *We say that a vector bundle F is a subbundle of the vector bundle E if it is a subsheaf and the inclusion morphism $F \hookrightarrow E$ is a vector bundle morphism.*

Intuitively, a vector bundle can be seen as a collection of vector spaces "glued" together. Thus, it is natural to try to extend some of the basic "operations" defined on vector spaces to vector bundles.

Proposition 1.4.4. *Let $f : (X, E, p) \to (X, F, q)$ be a morphism of vector bundles over X, i.e., such that $\mathrm{rk}(f_x)$ is locally constant. Consider the subfiber spaces $\mathrm{Ker}(f) \subseteq E$ and $\mathrm{Im}(f) \subseteq F$ defined by the restriction of p to the subspace $\bigcup_{x\in X} \mathrm{Ker}(f_x) = \mathrm{Ker}(f)$ and the restriction of q to the subspace $\bigcup_{x\in X} \mathrm{Im}(f_x) = \mathrm{Im}(f)$ respectively. Then the fibre spaces $\mathrm{Ker}(f)$ and $\mathrm{Im}(f)$ are vector bundles.*

Proof. The restriction of p and q to $\mathrm{Ker}(f)$ and $\mathrm{Im}(f)$ respectively clearly give them a fibre space structure whose fibers are vector spaces. In order to show that these fibre spaces are vector bundles we only have to find local trivializations given by fibrewise linear functions.

For any $x \in X$ let $U \subseteq X$ be an open neighborhood of x where E and F are trivial and such that f_y has constant rank r' on U.

For every trivialization of E and F on U, f can be represented as a morphism $g : U \times \mathbb{A}^n \to U \times \mathbb{A}^m$ which has the form $g(y, v) = (y, T(y)v)$ for a certain morphism $T : U \to \mathrm{Mat}(k, m, n)$. For every $y \in U$ $\mathrm{rk}(T(y)) = r'$, so we can choose local trivializations for E and F on U such that T has the form

$$T(y) = \begin{pmatrix} A(y) & 0 \\ 0 & 0 \end{pmatrix}$$

for every $y \in U$, where A is a morphism $A : U \to \mathrm{Mat}(k, r')$. Taking a narrower neighborhood if necessary, we can further assume that $A : U \to \mathrm{GL}(k, r')$, as $\mathrm{GL}(k, r')$ is open in $\mathrm{Mat}(k, r')$. Let $\pi : U \times \mathbb{A}^n \to U \times \mathbb{A}^{n-r'}$ be the projection into the second factor. Since A is invertible, the restriction $\pi|_{\mathrm{Ker}(w)}$ is an isomorphism $\mathrm{Ker}(w) \to U \times \mathbb{A}^{n-r'}$. Hence, $\mathrm{Ker}(g)$ is locally trivial and thus a vector bundle. Let g^t be the morphism given by the transpose matrix $A(y)^t$ and the formula $g^t(y, v) = (y, A(y)^t v)$. Then $\mathrm{Im}(g) = \mathrm{Ker}(g^t)$ is also a vector bundle. □

In an analogous way of proposition 1.4.4, it can be proved that the quotient of a vector bundle by a subbundle is a vector bundle. In general, the quotient of a locally free sheaf by a subsheaf is not necessarily a locally free sheaf. Let $i : F \to E$ be the inclusion morphism of a locally free subsheaf F into a locally free sheaf E over X. If $x \in X$ is a point such as $\mathrm{rk}\, f$ changes, then it can be proved that the sheaf E/F will have a torsion element in a neighborhood of x. This leads up to the following theorem

Proposition 1.4.5. *Let E, F be locally free sheaves such that F is a subsheaf of E. Let $i : E \to F$ be inclusion morphism. Then i is a vector bundle morphism if and only if E/F is torsion free.*

If F is a subsheaf of E that is not a subbundle, we would like to know if there exist an "extension" of F to a locally free sheaf $\overline{F}$ of the same rank such that F was a subsheaf of $\overline{F}$ and $\overline{F}$ was a subbundle of E. The following lemma combined with the previous proposition proves that this is always possible.

Lemma 1.4.6. *Let E be a locally free sheaf of rank r and E' be a subsheaf of rank r'. Then there exist a subsheaf $\overline{E'}$ of rank r' containing E' such that $E/\overline{E'}$ is locally free.*

Proof. We have the following exact sequence

$$0 \to E' \to E \to E/E' \to 0$$

Let T be the torsion submodule of E/E'. If E/E' is torsion-free then it is enough to take $\overline{E'} = E'$. Otherwise,$T \neq 0$. Let us consider the torsion-free quotient sheaf $(E/E')/T$. Let $\pi : E/E' \to (E/E')/T$ be the canonical projection. We define $\overline{E'}$ to be the kernel of the composition map $E \to E/E' \to (E/E')/T$. Then we have the following exact commutative diagram

$$
\begin{tikzcd}
 & & & 0 \arrow[d] & \\
 & 0 \arrow[d] & & T \arrow[d] & \\
0 \arrow[r] & E' \arrow[r] \arrow[d, "j"] & E \arrow[r] \arrow[d, "\mathrm{Id}"] & E/E' \arrow[r] \arrow[d, "\pi"] & 0 \\
0 \arrow[r] & \overline{E'} \arrow[r] \arrow[d] & E \arrow[r] & (E/E')/T \arrow[r] \arrow[d] & 0 \\
 & \overline{E'}/E' \arrow[d] & & 0 & \\
 & 0 & & &
\end{tikzcd}
\tag{1.4.1}
$$

As $E/\overline{E'}$ is torsion free, $\overline{E'}$ is a subbundle of E. The snake lemma then states that the sequence

$$\operatorname{Ker}(j) \to \operatorname{Ker}(\mathrm{Id}) \to \operatorname{Ker}(\pi) \to \operatorname{Coker}(j) \to \operatorname{Coker}(\mathrm{Id}) \to \operatorname{Coker}(\pi)$$

is exact. As $\operatorname{Ker}(\mathrm{Id}) = \operatorname{Coker}(\mathrm{Id}) = 0$, we get the exact sequence

$$0 \to \operatorname{Ker}(\pi) = T \to \operatorname{Coker}(j) = \overline{E'}/E' \to 0$$

Therefore, $\overline{E'}/E' \cong T$. As the sheaf T is a torsion sheaf, $\overline{E'}$ and E' have the same rank. □

We call the subbundle $\overline{E'}$ the saturation of the subsheaf E'. One of its most important properties is the following

Proposition 1.4.7. *Let E' be a subsheaf of a vector bundle E over a curve X. Let $\overline{E'}$ be the saturation of E'. Then* $\deg(\overline{E'}) \geq \deg(E')$.

Proof. As a consequence of the proof of the last proposition, we have a short exact sequence

$$0 \to E' \to \overline{E'} \to T \to 0$$

where T is the torsion sheaf of E/E'. By definition of the degree for torsion sheafs given in 1.3.12 we get that

$$\deg(\overline{E'}) = \deg(E') + \deg(T)$$

As X is a curve, equation (1.3.2) gives us that $\deg(T) = h^0(X,T) \geq 0$. Then

$$\deg(\overline{E'}) = \deg(E') + h^0(X,T) \geq \deg(E')$$

□

Chapter 2

Moduli Spaces

There are many problems in algebraic geometry, specially classification problems, in which we need to study a certain collection of geometric objects $\mathcal{A}$ with a given equivalence relation $\sim$. Some natural examples include

- The set of linear subspaces of $\mathbb{A}^n$
- The set of subvarieties of a given algebraic variety X
- The set of holomorphic vector bundles over a certain algebraic variety X with the equivalence relation given by bundle isomorphism

There usually exist a notion of "continuity" on the set of equivalence classes $\mathcal{A}/\sim$ and we would like to parametrize this set in a way which allows us to give it an algebro-geometric structure. Ideally, we would want to give $\mathcal{A}/\sim$ a variety structure, but this is not always going to be possible. Moduli spaces give a way to obtain this kind of algebraic structure.

In this chapter we will introduce different related formalisms for the concept of moduli space. We will analyze as well some of the main geometric examples of moduli spaces and we will introduce the tools needed. The main references are [HL96] and [New78].

Let k be a fixed algebraically closed field. In order to simplify the notation, from know on we will refer to schemes over k simply as schemes. We will denote the category of schemes over k as Sch.

2.1 Parametric families and moduli problems

In order to fix the algebraic structure of a moduli space we first have to fix a way for the the elements of $\mathcal{A}$ to be parametrized by a scheme. These parametrizations will be specified by a notion of *family* of elements of $\mathcal{A}$. The notion of family and an appropriate notion of equivalence relation among families will be the base for what we will call a *moduli problem*.

The intuitive idea of a family of objects of $\mathcal{A}$ parametrized by a scheme T is that we have a certain collection X_T such that for each $t \in T$ we have an object X_t and the objects are "glued together" according to the structure in T.

For example, given a variety X, taking $\mathcal{A}$ as the set of vector bundles over X with the equivalence relation given by bundle isomorphism, a family of objects of $\mathcal{A}$ parametrized by T is a vector bundle $\mathcal{E}$ over $X \times T$. The objects $\mathcal{E}_t$ correspond to the restriction of $\mathcal{E}$ to $X \times \{t\}$.

$$\begin{tikzcd} \mathcal{E}_t \arrow[r, hook] \arrow[d] & \mathcal{E} \arrow[d] \\ X \times \{t\} \arrow[r, hook] & X \times T \end{tikzcd} \tag{2.1.1}$$

As the original collection $\mathcal{A}$ has an equivalence relation $\sim$ we would like to extend it to an equivalence relation between families. This turns out to be a critical point in the definition of the moduli problem, as it will determine how the algebraic structure interacts with the equivalence classes. In the previous example a first attempt could be to consider two families of vector bundles $\mathcal{E}$ and $\mathcal{E}'$ parametrized by T as equivalent if $\mathcal{E}$ and $\mathcal{E}'$ were isomorphic as vector bundles over $X \times T$. Nevertheless, this equivalence relations turns out to be too weak.

Let us consider a line bundle over T, $L \to T$. Let $\pi : X \times T \to T$ be the projection. Let $\mathcal{E}$ be a family of vector bundles parametrized by T. Then $\mathcal{E}' := \mathcal{E} \otimes \pi^* L$ is another family parametrized by T that will not be, in general, isomorphic as a vector bundle to $\mathcal{E}$ (for example, they will have different degree if L has nontrivial degree).

On the other hand, if we look at the collection of objects of the families $\mathcal{E}$ and $\mathcal{E}'$ we have that for every $t \in T$

$$\mathcal{E}_t = \mathcal{E}_t \otimes \mathbb{C} \cong \mathcal{E}_t \otimes (\pi^* L_t) \cong (\mathcal{E} \otimes \pi^* L)_t = \mathcal{E}'_t$$

In some way this behavior goes against the intuitive idea of when two families should be considered equivalent. We got two families parametrized by the same scheme that have pointwise isomorphic elements but that are nonisomorphic as families.

In order to expect a natural solution to a moduli problem we must impose some restrictions to what we can consider as a family.

Definition 2.1.1. *A notion of family of objects of $\mathcal{A}$ satisfies*

a) A family parametrized by a point is a single object of $\mathcal{A}$.

b) There is a notion of equivalence of families parametrized by any given variety T, which reduces to the given equivalence relation $\sim$ on A when T consists of a single point. We will denote this relation again, abusing notation, by $\sim$.

c) For any morphism $\varphi : S \to T$ and any family X_T parametrized by T there exist a notion of pullback of families that allows us to define a family $\varphi^* X_T$ parametrized by S. Moreover if R, S, T are schemes and $\varphi : R \to S$ and $\psi : S \to T$ are morphisms, the pullback satisfies

$$(\psi \circ \varphi)^* = \varphi^* \circ \psi^* \quad \textit{and} \quad \mathrm{id}_T^* X_T = X_T \quad . \tag{2.1.2}$$

d) *The pullback is compatible with* $\sim$ *in the sense that if* $\varphi : T' \to T$ *is a morphism and* X, X' *are two families of objects of* $\mathcal{A}$ *parametrized by* T *such that* $X \sim X'$ *then* $\varphi^* X \sim \varphi^* X'$.

Conditions (a), (b) and (d) are simply compatibility conditions between the different notions that will be considered in the problem. The concept that really fixes the structure of the families is the notion of pullback and the functorial properties given in (c).

For example, let us fix a notion of pullback and a notion of equivalence of classes compatible with $\sim$. Let's take an arbitrary scheme T and a point $t \in T$, and consider the inclusion $i : \{t\} \to T$. For every family X_T over T, one may take the pullback by i and get a family $X_t := i^* X_T$ over $\{t\}$ by (c). Condition (a) implies that X_t is, in fact, a single object of $\mathcal{A}$, so we have recovered the initial intuitive idea that X_T is a collection of elements $X_t \in \mathcal{A}$ for every $t \in T$.

In addition, the compatibility condition (d) tells us that if $X_T \sim X'_T$ for two families over T, then for every $t \in T$ we have $X_t = i^* X_T \sim i^* X'_T = X'_t$, so every class of families $[X_T]$ over T can be considered as a collection of equivalence classes $[X_t] \in \mathcal{A}/\sim$ for every $t \in T$.

Pullback properties (2.1.2) codifies the fact that the operation of taking the possible families for a given scheme T is a contravariant functor from the category of schemes to the category of sets. For every scheme T let $\mathcal{F}(T)$ be the set of equivalence classes of families parametrized by T. For every morphism $\varphi : T' \to T$ let $\mathcal{F}(\varphi) = \varphi^* : \mathcal{F}(T) \to \mathcal{F}(T')$. Conditions (2.1.2) imply that $\mathcal{F} : \mathrm{Sch} \to \mathrm{Sets}$ is a contravariant functor. We will revisit this later, as this functorial point of view is the key for the different definitions of moduli spaces that we will consider.

Once we have formalized the requirements that an admissible notion of family must satisfy we can integrate all the information needed to define the algebraic structure of the moduli into what we will call a *moduli problem.*

Definition 2.1.2. *A* moduli problem *is*

a) *A notion of objects* $\mathcal{A}$ *and an equivalence relation between elements of* $\mathcal{A}$.

b) *A notion of family parametrized by a scheme and equivalence of families.*

c) *A notion of pullback of families compatible with equivalence.*

Since the conditions for a notion of family to be admissible for a moduli problem are just compatibility statements between the notions of the problem, it is clear that there may exist different notions of family for the same class space $\mathcal{A}/\sim$.

In the example of vector bundles seen before, we could have taken trivial families, defining a family of vector bundles parametrized by a scheme T to be the trivial family $X_T = E$ for some vector bundle $E \to X$, with the pullback given by the identity and taking two families as equivalent if the corresponding vector bundles are so. This is obviously a well defined notion of family and one can build a corresponding moduli problem from it. Of course, the corresponding moduli space - in case it exists - would be of no interest, as we would have not provided any structure elements that allow us to "glue" the vector bundles.

On the other hand, as we saw in the example, for the same class space $\mathcal{A}/\sim$ and the same notion of family there may be multiple possible choices for the equivalence relation for families. As we will see later on, the properties of this equivalence relation and its interaction with the pullback of families will be crucial in order to determine the kind of solution one might expect from a moduli problem.

In a categorical language, we see that the functor $\mathcal{F} : \mathrm{Sch} \to \mathrm{Sets}$ essentially has all the information of the moduli problem needed to define the structure of the moduli space.

- The set $\mathcal{A}/\sim$ can be retrieved from the image of $\{pt\}$.
- The notion of class of family modulo the given equivalence relation for families is made explicit for every scheme by the image of the functor.
- The pullback definition is naturally given by the morphism transformations defined by the functor.

Then, we could alternately define a moduli problem to be a contravariant functor $\mathcal{F} : \mathrm{Sch} \to \mathrm{Sets}$. While this definition doesn't explicitly give information about the equivalence relation and the set $\mathcal{A}$, we will see that it contains enough information about the structure of the set $\mathcal{A}/\sim$ and the classes of families to define the moduli space.

2.2 Moduli spaces

Given a moduli problem we want to build a moduli space that "solves" the problem by giving an algebraic structure to the set $\mathcal{A}/\sim$ that reflects the structure of all the families that parametrize objects in $\mathcal{A}$ in a way compatible with the equivalence relation defined between families.

Let $\mathcal{M}$ be a scheme whose underlying set is $\mathcal{A}/\sim$. For every class of families $[X]$ parametrized by T we have a map $\nu_{[X]} : T \to \mathcal{M}$ given by $\nu_{[X]}(t) = [X_t]$. As we saw before, this map is well defined as a morphism of sets. If we want $\mathcal{M}$ to reflect the algebraic structure of the families, it would be natural to ask maps $\nu_{[X]}$ to be morphisms of schemes for all classes of families $[X]$. This idea can be expressed in the language of categories.

For every $T \in \mathrm{Sch}$, let us define $\Phi(T) : \mathcal{F}(T) \to \mathrm{Hom}_{\mathrm{Sets}}(T, \mathcal{M})$ given by

$$\Phi(T)([X]) = \nu_{[X]} \quad .$$

Conditions (2.1.2) imply that Φ determines a natural transformation

$$\Phi : \mathcal{F} \to \mathrm{Hom}_{\mathrm{Sets}}(-, \mathcal{M}) \quad .$$

The notion of $\nu_{[X]}$ being an algebraic morphism can be stated asking Φ to factorize as a natural transformation

$$\Phi : \mathcal{F} \to \mathrm{Hom}_{\mathrm{Sch}}(-, \mathcal{M}) \quad .$$

As we want $\mathcal{M}$ to represent precisely the structure of the moduli problem it is natural to further ask Φ to be a functor isomorphism, that is, that the functor $\mathcal{F}$ is represented by the pair $(\mathcal{M}, \Phi)$. This leads up to the following definition

Definition 2.2.1. *A fine moduli space for a given moduli problem is a pair $(\mathcal{M}, \Phi)$ which represents the functor $\mathcal{F}$ in the category of schemes, i.e., $\mathcal{M}$ is a scheme and Φ is an isomorphism between $\mathcal{F}$ and $\mathrm{Hom}_{\mathrm{Sch}}(-, \mathcal{M})$.*

From the definition of fine moduli space it's not clear a priori whether the scheme $\mathcal{M}$ has $\mathcal{A}/\sim$ as base set, neither the structure of the isomorphism Φ. Nevertheless, both things can be derived from the structure of the functor $\mathcal{F}$.

Proposition 2.2.2. *Let $(\mathcal{M}, \Phi)$ be a fine moduli space for the functor $\mathcal{F}$. Then there exist a bijection $\psi : \mathcal{A}/\sim \to \mathcal{M}$ such that for every $T \in \mathrm{Sch}$ and every $[X] \in \mathcal{F}(T)$,*

$$\Phi(T)([X]) = \psi \circ \nu_{[X]}$$

Proof. As Φ is an isomorphism, we have a bijection

$$\Phi(\{pt\}) : \mathcal{A}/\sim = \mathcal{F}(\{pt\}) \to \mathrm{Hom}_{Sch}(\{pt\}, \mathcal{M})$$

Let $\rho : \mathrm{Hom}(\{pt\}, \mathcal{M}) \xrightarrow{\cong} \mathcal{M}$ be the evaluation morphism $\varphi \mapsto \varphi(pt)$. Then the desired bijection is $\psi = \rho \circ \Phi(\{pt\})$. Let $T \in \mathrm{Sch}$ be any scheme and let $[X] \in \mathcal{F}(T)$ be any family parametrized by T. For every $t \in T$ let us consider the inclusion $i : \{t\} \to T$. By definition of the functor $\mathcal{F}$, $\mathcal{F}(i) = i^*$. On the other hand, for every $[X] \in \mathcal{F}(T)$, we have defined

$$\mu_{[X]}(t) = [X_t] = i^*[X] \quad .$$

Then, as Φ is a natural transformation, it induces a commutative diagram

$$\begin{tikzcd}
\mathcal{F}(T) \arrow[r, "\Phi(T)"] \arrow[d, "i^*"'] & \mathrm{Hom}_{\mathrm{Sch}}(T, \mathcal{M}) \arrow[d, "i^\sharp"', "\varphi \mapsto \varphi \circ i"] \\
\mathcal{F}(\{pt\}) \arrow[r, "\Phi(\{pt\})"] \arrow[dr, "\psi"'] & \mathrm{Hom}_{\mathrm{Sch}}(\{pt\}, \mathcal{M}) \arrow[d, "\rho"] \\
 & \mathcal{M}
\end{tikzcd} \tag{2.2.1}$$

Consequently we have

$$\Phi(T)([X])(t) = (\rho \circ i^\sharp \circ \Phi(T))([X]) = (\psi \circ i^*)([X]) = \psi(\nu_{[X]}(t)) = (\psi \circ \nu_{[X]})(t)$$

As this is true for every $t \in T$, we have that $\Phi(T)([X]) = \psi \circ \nu_{[X]}$. □

Another possible question is whether the scheme $\mathcal{M}$ of a fine moduli space is unique (up to isomorphism).

Proposition 2.2.3. *If $(\mathcal{M}, \Phi)$ and $(\mathcal{M}', \Phi')$ are two fine moduli spaces for the functor $\mathcal{F}$ then $\mathcal{M} \cong \mathcal{N}$.*

Proof. By definition of fine moduli space, $\alpha := \Phi' \circ \Phi^{-1} : \mathrm{Hom}_{\mathrm{Sch}}(-, \mathcal{M}) \to \mathrm{Hom}_{\mathrm{Sch}}(-, \mathcal{M}')$ is a isomorphism of functors with inverse $\beta := \Phi \circ (\Phi')^{-1}$, i.e., α and β are natural transformations such that $\alpha \circ \beta = \mathrm{Id}_{\mathrm{Hom}_{\mathrm{Sch}}(-,\mathcal{M})}$ and $\beta \circ \alpha = \mathrm{Id}_{\mathrm{Hom}_{\mathrm{Sch}}(-,\mathcal{M}')}$. By Yoneda's lemma

$$\mathrm{Hom}(\mathrm{Hom}_{\mathrm{Sch}}(-, \mathcal{M}), \mathrm{Hom}_{\mathrm{Sch}}(-, \mathcal{M}')) \cong \mathrm{Hom}_{\mathrm{Sch}}(\mathcal{M}', \mathcal{M})$$

and

$$\mathrm{Hom}(\mathrm{Hom}_{\mathrm{Sch}}(-, \mathcal{M}'), \mathrm{Hom}_{\mathrm{Sch}}(-, \mathcal{M})) \cong \mathrm{Hom}_{\mathrm{Sch}}(\mathcal{M}, \mathcal{M}')$$

and there exist morphisms $f \in \mathrm{Hom}_{\mathrm{Sch}}(\mathcal{M}, \mathcal{M}')$ and $g \in \mathrm{Hom}_{\mathrm{Sch}}(\mathcal{M}', \mathcal{M})$ such that for every scheme T and every morphism $u \in \mathrm{Hom}_{\mathrm{Sch}}(T, \mathcal{M})$

$$\alpha(T)(u) = f \circ u$$

and for every $v \in \mathrm{Hom}_{\mathrm{Sch}}(T, \mathcal{M}')$

$$\beta(T)(v) = g \circ v$$

Hence, $(\alpha \circ \beta)(\mathcal{M}')$ maps $\mathrm{Id}_{\mathcal{M}'}$ to $f \circ g$ and $(\beta \circ \alpha)(\mathcal{M})$ maps $\mathrm{Id}_{\mathcal{M}}$ to $g \circ f$. Since $\alpha \circ \beta$ and $\beta \circ \alpha$ are the identity functors then $f \circ g = \mathrm{Id}_{\mathcal{M}'}$ and $g \circ f = \mathrm{Id}_{\mathcal{M}}$, so $f : \mathcal{M} \to \mathcal{M}'$ is a scheme isomorphism. ☐

Moreover, if $(\mathcal{M}, \Phi)$ is a fine moduli space, we can take a family $[U] := \Phi(\mathcal{M})^{-1}(\mathrm{Id}_{\mathcal{M}})$ parametrized by $\mathcal{M}$. We will call U a universal family for the problem. Let T be any scheme and $[X]$ any class of families parametrized by T. Let $f := \Phi(T)([X]) : S \to M$. Then by definition of Φ the following diagram commutes

$$\begin{tikzcd}
{[U]} \arrow[rrr, mapsto] \arrow[ddd, mapsto] & & & \mathrm{id}_{\mathcal{M}} \arrow[dddd, mapsto, "f^\sharp"'] \\
 & & & \\
 & \mathcal{F}(\mathcal{M}) \arrow[r, "\Phi(\mathcal{M})"] \arrow[d, "f^*"'] & \mathrm{Hom}_{\mathrm{Sch}}(\mathcal{M}, \mathcal{M}) \arrow[d, "f^\sharp"', "\varphi \mapsto \varphi \circ f"] & \\
f^*[U] & \mathcal{F}(T) \arrow[r, "\Phi(T)"] & \mathrm{Hom}_{\mathrm{Sch}}(T, \mathcal{M}) & \\
 & {[X]} \arrow[rr, mapsto] & & f
\end{tikzcd} \tag{2.2.2}$$

The commutativity of the diagram implies that $f^*[U] = [X]$. As this is true for every scheme T and every class of families $[X]$ parametrized by T, we conclude that all the classes of families are the pullback of $[U]$ by a certain morphism $T \to M$.

On the other hand, let $f : S \to M$ be any other morphism in (2.2.2) such that $f^*[U] = [X]$. The diagram being commutative implies that

$$f = f^\sharp(\mathrm{id}_{\mathcal{M}}) = (f^\sharp \circ \Phi(\mathcal{M}))([U]) = (\Phi(T) \circ f^*)([U]) = \Phi(T)(f^*[U]) = \Phi(T)([X])$$

This way we have proved that for every class of families $[X]$ over T there exist an unique morphism $f : T \to \mathcal{M}$ such that $[X] = f^*[U]$.

This leads up to this equivalent definition of fine moduli space

Definition 2.2.4. *A fine moduli space is a pair consisting of a variety $\mathcal{M}$ and a universal class of families $[U]$ parametrized by $\mathcal{M}$, such that for every variety T there is a unique morphism $f : T \to \mathcal{M}$ such that $[X] = f^*[U]$.*

We have already proved that a fine moduli space in the sense of definition 2.2.1 is a fine moduli space in the sense of definition 2.2.4. Now let $[U]$ be a universal family for a moduli problem parametrized by $\mathcal{M}$. Let us prove that $\mathrm{Hom}_{\mathrm{Sch}}(-,\mathcal{M})$ represents $\mathcal{F}$.

It suffices to take, for every class of families $[X]$ parametrized by T, the unique morphism $\Phi(T)([X]) \in \mathrm{Hom}_{\mathrm{Sch}}(T,\mathcal{M})$ such that $[X] = \Phi(T)([X])^*[U]$. The uniqueness of such morphism implies that $\Phi(T) : \mathcal{F}(T) \to \mathrm{Hom}_{\mathrm{Sch}}(T,\mathcal{M})$ is a bijection with inverse given by the pullback of $[U]$ by the morphism.

To prove that Φ is an isomorphism it is enough to prove that Φ^{-1} is a natural transformation. Let S, T schemes and $f : S \to T$ a scheme morphism.

$$\begin{tikzcd}
\mathcal{F}(T) \arrow[r, "\Phi(T)"] \arrow[d, "f^*"'] & \mathrm{Hom}_{\mathrm{Sch}}(T,\mathcal{M}) \arrow[d, "f^\sharp"', "\varphi\mapsto\varphi\circ f"] \\
\mathcal{F}(S) \arrow[r, "\Phi(S)"] & \mathrm{Hom}_{\mathrm{Sch}}(S,\mathcal{M})
\end{tikzcd} \tag{2.2.3}$$

Let $[X]$ be any family parametrized by T. By definition of $\Phi(T)$, $[X] = \Phi(T)([X])^*[U]$, so $f^*[X] = (f^* \circ \Phi(T)([X])^*)[U]$. By condition (c) of definition 2.1.1,

$$f^*[X] = (f^* \circ \Phi(T)([X])^*)[U] = (\Phi(T)([X]) \circ f)^*[U] = (f^\sharp(\Phi(T)[X]))^*[U]$$

On the other hand, we have proved that $\Phi(S)(f^*[X])$ is the unique morphism such that $f^*[X] = \Phi(S)(f^*[X])^*[U]$, so we must have

$$(\Phi(S) \circ f^*)[X] = \Phi(S)(f^*[X]) = f^\sharp(\Phi(T)([X])) = (f^\sharp \circ \Phi(T))[X]$$

As this is true for every $[X]$, (2.2.3) is commutative, so Φ is an isomorphism between $\mathcal{F}$ and $\mathrm{Hom}_{\mathrm{Sch}}(-,\mathcal{M})$.

Remark 2.2.5. *If we fix a fine moduli space $(\mathcal{M},\varphi)$, there may exist more than one class of families over $\mathcal{M}$ that satisfy the universal property of definition 2.2.4.*

The given construction of a universal family for a fine moduli space takes the preimage of $\mathrm{id}_{\mathcal{M}}$, but we could have taken any automorphism of $\mathcal{M}$. Let $\varphi \in \mathrm{Aut}_{\mathrm{Sch}}(\mathcal{M})$, and let us define $[U_\varphi] := \Phi(T)^{-1}(\varphi)$. Let T be any scheme and $[X]$ any class of families parametrized by T. Let $f := \Phi(T)([X]) : S \to M$. Then by definition of Φ the following diagram commutes

$$
\begin{tikzcd}
{[U_\varphi]} \arrow[rrr, mapsto] \arrow[dd, mapsto] & & & \varphi \arrow[ddd, mapsto] \\
& \mathcal{F}(\mathcal{M}) \arrow[r, "\Phi(\mathcal{M})"] \arrow[d, "(\varphi^{-1}\circ f)^*"'] & \mathrm{Hom}_{\mathrm{Sch}}(\mathcal{M},\mathcal{M}) \arrow[d, "(\varphi^{-1}\circ f)^\sharp"', "\psi\mapsto\psi\circ\varphi^{-1}\circ f"] & \\
(\varphi^{-1}\circ f)^*[U_\phi] & \mathcal{F}(T) \arrow[r, "\Phi(T)"] & \mathrm{Hom}_{\mathrm{Sch}}(T,\mathcal{M}) & \\
& {[X]} \arrow[rr, mapsto] & & f
\end{tikzcd}
\tag{2.2.4}
$$

Proposition 2.2.6. *For a fixed fine moduli space $(\mathcal{M}, \varphi)$, there is a bijection between the set of equivalence classes of universal families over $\mathcal{M}$ and $\mathrm{Aut}_{\mathrm{Sch}}(\mathcal{M})$.*

Proof. Let $[V]$ be a universal family. As $[U]$ is a family parametrized by $\mathcal{M}$, by definition of universal family, there is a morphism $f : \mathcal{M} \to \mathcal{M}$ such that $[U] = f^*[V]$. On the other hand, as $[U]$ is itself a universal family, there exist a morphism $g : \mathcal{M} \to \mathcal{M}$ such that $[V] = g^*[U]$. Then we have $[U] = (f^* \circ g^*)[U] = (g \circ f)^*[U]$. As Φ is a natural transformation we have

$$
\begin{tikzcd}
\mathcal{F}(\mathcal{M}) \arrow[r, "\Phi(\mathcal{M})"] \arrow[d, "(g\circ f)^*"'] & \mathrm{Hom}_{\mathrm{Sch}}(\mathcal{M},\mathcal{M}) \arrow[d, "(g\circ f)^\sharp"] \\
\mathcal{F}(\mathcal{M}) \arrow[r, "\Phi(\mathcal{M})"] & \mathrm{Hom}_{\mathrm{Sch}}(\mathcal{M},\mathcal{M})
\end{tikzcd}
\tag{2.2.5}
$$

We have $\Phi(\mathcal{M})([U]) = \mathrm{id}_{\mathcal{M}}$ and $(g \circ f)^*[U] = [U]$, so $(g \circ f)^\sharp(\mathrm{id}_{\mathcal{M}}) = \mathrm{id}_{\mathcal{M}}$ and thus, $g \circ f = \mathrm{id}_{\mathcal{M}}$. In an analogous way, we have $f \circ g = \mathrm{id}_{\mathcal{M}}$, so $f, g \in \mathrm{Aut}_{\mathrm{Sch}}(\mathcal{M})$. We also proved that $f = \Phi(\mathcal{M})([V])$, and $\Phi(\mathcal{M})$ is bijective, so $\Phi(\mathcal{M})$ is an injective morphism that sends every class of a universal family to an automorphism. We previously proved that for every automorphism $\varphi \in \mathrm{Aut}_{\mathrm{Sch}}(\mathcal{M})$, $\Phi(\mathcal{M})^{-1}(\varphi)$ is the class of a universal family, so $\Phi(\mathcal{M})$ restricted to the subset of universal families is the desired bijection. Moreover, as Φ is a natural transformation, $\Phi(\mathcal{M})$ turns out to be functorial when restricted to classes of universal families. $\square$

Therefore, we can obtain all the possible universal families over $\mathcal{M}$ as the pullback of the canonical universal families $\Phi(\mathcal{M})^{-1}(\mathrm{Id}_{\mathcal{M}})$ by a transformation of the base scheme $\mathcal{M}$. As we have proved that all the possible fine moduli spaces definable for a certain moduli problem differ in an isomorphism of the space, we can think of all the non-canonical universal families of $\mathcal{M}$ as the canonical universal family for another solution of the moduli problem $\mathcal{M}' \cong \mathcal{M}$. In this way, we can say that the universal family is unique up to an isomorphism of the moduli space.

As an example of the notion of universal family, in the case of families of vector bundles described in the previous section, a universal family for the problem would be a vector bundle $\mathcal{E} \to X \times \mathcal{M}$ such that for every other vector bundle $\mathcal{E}' \to X \times T$

there exist a morphism $f : T \to \mathcal{M}$ and a line bundle $L \to T$, such that $\mathcal{E}' \cong f^*\mathcal{E} \otimes \pi^* L$.

Unfortunately, for a general variety X, there does not exist such vector bundle. We will see that one must impose additional conditions to the vector bundles and select the equivalence relation for families carefully in order to obtain a moduli space.

As a clear example of nonexistence of a fine moduli space, let us consider the moduli problem of one-dimensional complex vector spaces up to vector space isomorphism. We will define a family of one-dimensional vector spaces parametrized by a scheme T to be a line bundle over T, taking line bundle isomorphism as the equivalence relation between families. We want to build a fine moduli space $(\mathcal{M}, \Phi)$ for this problem. As every one-dimensional vector space is isomorphic to $\mathbb{C}$, the set $\mathcal{A}/\sim$ is a single point. If it existed a fine moduli space, its underlying set would be $\mathcal{A}/\sim$, so $\mathcal{M}$ would be a single point. We will see that this is not possible.

Let us take $T = S^1$ as the parameter space. There exist at least two non-isomorphic line bundles over S^1, the cylinder and the Mbius band. Since the moduli space is a point, there exist a unique map $S^1 \to \mathcal{M}$, but this is impossible, since $\Phi(S^1) : \mathcal{F}(S^1) \to \mathrm{Hom}_{\mathrm{Sch}}(S^1, \mathcal{M})$ is bijective.

One of the main general obstructions is the existence of nontrivial automorphisms for families. As we saw in the previous example, if we have two nonisomorphic families that induce the same morphism to $\mathcal{M}$ there will not exist a fine moduli space for the problem. The existence of nontrivial automorphisms usually help that situation, as the actions of the automorphisms may not affect the morphism to $\mathcal{M}$ but will generate nonisomorphic families.

It is therefore necessary to find weaker conditions that allows us to determine a unique algebro-geometric structure for $\mathcal{M}$ without the need of a universal family. We will prove that weakening that universal property is enough for the existence of a compatible algebraic structure.

Definition 2.2.7. *A coarse moduli space for a given moduli problem is a scheme $\mathcal{M}$ together with a natural transformation $\Phi : \mathcal{F} \to \mathrm{Hom}_{\mathrm{Sch}}(-, \mathcal{M})$ such that*

a) $\Phi(\{pt\})$ *is bijective.*

b) For any variety $\mathcal{N}$ and any natural transformation $\psi : \mathcal{F} \to \mathrm{Hom}_{\mathrm{Sch}}(-, \mathcal{N})$, there exist a unique natural transformation

$$\Omega : \mathrm{Hom}_{\mathrm{Sch}}(-, \mathcal{M}) \to \mathrm{Hom}_{\mathrm{Sch}}(-, \mathcal{N})$$

such that $\psi = \Omega \circ \Phi$.

The main idea behind this definition is that in the case of fine moduli spaces, we proved that the definition implied a universal property for the variety $\mathcal{M}$ and a family parametrized by it. On the other hand, coarse moduli spaces impose a universal condition on the pair $(\mathcal{M}, \Phi)$.

Apart from the universal property, asking $\Phi(\{pt\})$ to be bijective is essential if we want $\mathcal{M}$ to be a variety over $\mathcal{A}/\sim$. By explicitly asking for it, we are in fact fixing that Φ has essentially the same structure of that of a fine moduli space.

Proposition 2.2.8. *Let $(\mathcal{M}, \Phi)$ be a coarse moduli space for the functor $\mathcal{F}$. Then there exist a bijection $\psi : \mathcal{A}/\sim \to \mathcal{M}$ such that for every $T \in$ Sch and every $[X] \in \mathcal{F}(T)$,*

$$\Phi(T)([X]) = \psi \circ \nu_{[X]}$$

Proof. The proof is completely analogous to that of proposition 2.2.2, as in that proof we only used that Ψ is a natural transformation, not an isomorphism, and we are explicitly asking $\Phi(\{pt\})$ to be bijective, so $\psi = \rho \circ \Phi(\{pt\})$ is the desired bijection. □

Proposition 2.2.9. *A coarse moduli space $(\mathcal{M}, \Phi)$ is a fine moduli space if and only if*

a) there exist a family U parametrized by $\mathcal{M}$ such that for all $m \in \mathcal{M}$, $[U_m] = \Phi(\{pt\})^{-1}(m)$, and

b) for any families X, X' parametrized by T, $\nu_{[X]} = \nu_{[X']}$ if an only if $X \sim X'$.

Proof. Let Ψ be the bijection defined in proposition 2.2.8. Let's prove that Φ is injective, i.e., that for every scheme T, $\Phi(T)$ is injective. Let $[X], [X']$ be classes of families parametrized by T such that $\Phi(T)([X]) = \Phi(T)([X'])$. By (a) we have $(\psi^{-1} \circ \Phi(T))([X]) = (\psi^{-1} \circ \Phi(T))([X'])$. By proposition 2.2.8, we get

$$\nu_{[X]} = (\psi^{-1} \circ \Phi(T))([X]) = (\psi^{-1} \circ \Phi(T))([X']) = \nu_{[X']}$$

By condition (b), we have $[X] = [X']$. Reciprocally, if $(\mathcal{M}, \Phi)$ is a coarse moduli space, $\Phi(T)$ is bijective, so we get (b).

If $(\mathcal{M}, \Phi)$ is a fine moduli space, the universal family clearly satisfies (a). Suppose that a coarse moduli space satisfies (a) and (b). Let us prove that $\Phi(T)$ is surjective for every scheme T. We may first prove that $\Phi(M)([U]) = \mathrm{id}_{\mathcal{M}}$. As Φ is a natural, for very $m \in \mathcal{M}$, taking $i : \{m\} \hookrightarrow \mathcal{M}$ diagram (2.2.1) commutes for every $m \in \mathcal{M}$.

Then we for all $m \in \mathcal{M}$ have $m = (\psi \circ i^*)([U]) = (\rho \circ i^\sharp \circ \Phi(M))([U]) = \Phi(M)([U])(m)$, so $\Phi(M)([U]) = \mathrm{id}_{\mathcal{M}}$.

On the other hand, let $f : T \to \mathcal{M}$ be any scheme morphism. As Φ is a natural transformation, the following diagram commutes

$$\begin{tikzcd}
\mathcal{F}(\mathcal{M}) \arrow[r, "\Phi(\mathcal{M})"] \arrow[d, "f^*"'] & \mathrm{Hom}_{\mathrm{Sch}}(\mathcal{M}, \mathcal{M}) \arrow[d, "f^\sharp"', "\varphi \mapsto \varphi \circ f"] \\
\mathcal{F}(T) \arrow[r, "\Phi(T)"] & \mathrm{Hom}_{\mathrm{Sch}}(T, \mathcal{M})
\end{tikzcd} \tag{2.2.6}$$

Thus, we get that

$$\Phi(T)(f^*[U]) = f^\sharp(\Phi(M)([U])) = f^\sharp(\mathrm{id}_{\mathcal{M}}) = f$$

This last equation tells us,that, in fact, the provided family $[U]$ turns out to be a universal family for the moduli problem. □

One of the main consequences of the previous proposition is that the conditions for a coarse moduli space to become a fine moduli space depend only on the notion of the equivalence relation of families. For example, condition (b) of the proposition can be always satisfied independently of the problem taking two families X, X' parametrized by T to be equivalent whenever $X_t \sim X'_t$ for all $t \in T$.

As an example, we can consider the moduli problem of classification of vector bundles. We have seen two possible ways of defining the notion of equivalence of classes. If E and E' are families parametrized by a scheme T,

a) $E \sim E'$ if they are isomorphic as vector bundles.

b) $E \sim E'$ if there exist a line bundle $L \to T$ such that $E' \cong E \otimes \pi^* L$.

We have already proved that if we take the equivalence relation (a) as the notion of equivalence of families, in general there will exist a line bundle $L \to T$ such that $E \ncong E \otimes \pi^* L = E'$. Nevertheless, for every $t \in T$, $E_t \cong E'_t$. Therefore, $\nu_{[E]} = \nu_{[E']}$, but $[E] \neq [E']$. This contradicts hypothesis (b) of proposition 2.2.9, so the corresponding moduli space, in the case if exists, would not be fine.

On the other hand, in the last section of this chapter we will see that under certain extra hypothesis (imposing stability conditions on the bundles and taking coprime rank and degree), the moduli problem built with the equivalence relation (b) is fine.

2.3 Line bundle moduli spaces

We have previously introduced the moduli problem for vector bundles over a curve. In this section we will study the simpler, yet complete case of the moduli problem of line bundles over a curve. From now on, X will denote a nonsingular complex algebraic curve of genus g unless it is stated the opposite. Let us consider $\mathcal{A}$ to be the set of line bundles over X with fixed degree $d \in \mathbb{Z}$. We will say that two line bundles are equivalent if they are isomorphic as vector bundles.

We will define a family of line bundles parametrized by a scheme T to be a line bundle $\mathcal{L} \to X \times T$. We will say that two families of line bundles parametrized by T, $\mathcal{L} \to X \times T$ and $\mathcal{L}' \to X \times T$ are equivalent if there exist a line bundle $L \to T$ such that $\mathcal{L}$ and $\mathcal{L}' \otimes \pi^* L$ are isomorphic as vector bundles. This equivalence relation clearly agrees with equivalence of line bundles over X whenever T is a single point. We can think of this equivalence relation as taking the quotient of the set of isomorphism classes of families of vector bundles by the trivial families, i.e., families that are trivial over X and thus, reduce to pullbacks of line bundles over T.

The pullback of families will be taken as the induced the pullback of a rank one free sheaf by a scheme morphism. As sets, if $f : S \to T$ is a morphism, it induces a morphism $\tilde{f} : X \times S \to X \times T$ taking $\tilde{f}(x, s) = (x, f(s))$. If $\mathcal{L} \to X \times T$ is a family, we define $f^* \mathcal{L} := \tilde{f}^* \mathcal{L}$.

As bundle isomorphisms are preserved by bundle pullback, the notion of pullback of families and the notion of equivalence relation of families are compatible according to definition 2.1.1, so the previously stated notions form a well posed moduli problem.

2.3.1 Picard variety

The categorical formulation of the previous moduli problem can be stated in a more geometrical way. Let Pic : Sch $\to$ Group be the functor that sends each scheme S to the group $H^1(S, \mathcal{O}_X^\times)$ of isomorphism classes of invertible sheaves on S. For very $d \in \mathbb{Z}$, we define the functor P_X^d : Sch $\to$ Sets taking every scheme T to

$$P_X^d(T) := \{\mathcal{L} \in \operatorname{Pic}(X \times T) | \forall t \in T \deg(\mathcal{L}_t) = d\} / \pi^* \operatorname{Pic}(T)$$

As the pullback acts trivially on the fibers $\mathcal{L}_t$, the degree is invariant through pullback of families, so to see that this is a well defined functor we only have to observe that the degree of a line bundle $\mathcal{L}_t$ is invariant under the action of $\pi^* \operatorname{Pic}(T)$. Let $L \in \operatorname{Pic}(T)$. For every $t \in T$, $(q^* L)_t \cong q^* L_t \cong \mathcal{O}_X$, so for every $t \in T$

$$\deg((\mathcal{L} \otimes \pi^* L)_t) = \deg(\mathcal{L}_t) + \deg(\pi^* L_t) = \deg(\mathcal{L}_t) + \deg(\mathcal{O}_X) = \deg(\mathcal{L}_t)$$

Therefore, for every $d \in \mathbb{Z}$, P_X^d is a well defined functor that clearly reflects completely the moduli problem.

We could have proposed a similar moduli problem without fixing the degree of the line bundles. Let $P_X(T)$ be the functor sending every scheme T to

$$P_X(T) := \operatorname{Pic}(X \times T) / \pi^* \operatorname{Pic}(T)$$

We have proved that the degree is compatible with the equivalence relation and it is an algebraic invariant for the classes of isomorphic vector bundles. On the other side, for every connected $T \in$ Sch, [CS86][§7.4.2(b)] proves that if $\mathcal{L}$ is a line bundle over $X \times T$, then $\deg(\mathcal{L}_t)$ is independent of t and it's invariant under pullbacks, so the functor P_X decomposes into

$$P_X(T) = \coprod_{d \in \mathbb{Z}} P_X^d(T)$$

We will see that all the previous moduli problems have an associated moduli space. We define the Picard variety of X, $\operatorname{Pic}(X)$ to be the fine moduli space associated to P_X. For every $d \in \mathbb{Z}$ we will define the Picard variety of X of degree d, $\operatorname{Pic}^d(X)$ to be the fine moduli space associated to P_X^d. As X is connected, we have

$$\operatorname{Pic}(X) = \coprod_{d \in \mathbb{Z}} \operatorname{Pic}^d(X)$$

and, in fact, it can be proved that $\operatorname{Pic}^d(X)$ are precisely the connected components of $\operatorname{Pic}(X)$.

Let P be a rational point of X. For every $d \in \mathbb{Z}$, let $\mathcal{L}_d$ be the line bundle associated to the degree d divisor dP. Then for every scheme T, if $\pi' : X \times T \to X$, the morphism $\mathcal{L} \mapsto \mathcal{L} \otimes \pi'^* \mathcal{L}_d$ induces an isomorphism $P_X^0(T) \to P_X^d(T)$, so P_X^0 and P_X^d are isomorphic functors for all $d \in \mathbb{Z}$.

Thus, P_X^d is representable for all d if and only if P_X^0 is representable. Taking this into account, we shall restrict our study to the functor P_X^0 and the corresponding component of the Picard variety, $\operatorname{Pic}^0(X)$.

2.3.2 Jacobian variety

We define the Jacobian variety over X to be $\mathrm{Jac}(X) := \mathrm{Pic}^0(X)$. The following theorem tells us that there exist, in fact a solution for the moduli problem defined for the jacobian. In order to do so we will use the following type of divisors

Definition 2.3.1. *Let $\pi : X \to T$ be a scheme morphism. A relative effective Cartier divisor on X/T is a Cartier divisor on X that is flat over T when regarded as a subscheme of T.*

Theorem 2.3.2. *The functor P_X^0 is representable by a variety $\mathrm{Jac}(X)$.*

Proof. We shall only outline the proof for this theorem. A complete proof can be found in the chapter by J.S. Milne [CS86, §7].

Given a complete smooth curve X and a scheme T, for every $d \in \mathbb{Z}$, we define $\mathrm{Div}_X^d(T)$ to be the set of relative effective Cartier divisors on $X \times T/T$ of degree d. Proposition [CS86, §7.3.7] shows that $\mathrm{Div}_X^d : \mathrm{Sch} \to \mathrm{Sets}$ is a contravariant functor.

Let $X^{(d)}$ be the symmetric power of X, that is, the quotient $S_d \backslash V^r$, where S_d is the symmetric group on d letters. Theorem [CS86, §7. 3.13] proves that Div_X^d is isomorphic to $\mathrm{Hom}_{\mathrm{Sch}}(-, X^{(d)})$.

There exist a natural transformation $f : \mathrm{Div}_X^d \to P_X^d$ sending a relative effective Cartier divisor D on $X \times T/T$ to the corresponding line bundle.

As Div_X^d is representable, it can be proved that if there exist a section s to $f : \mathrm{Div}_X^d \to P_X^d$ then P_X^d is representable by a closed subscheme of $X^{(d)}$. If s is such section, let $\varphi = s \circ f : \mathrm{Div}_X^d \to \mathrm{Div}_X^d$. As Div_X^d is representable by $X^{(d)}$, φ is representable by a morphism of varieties that we shall call again, abusing the notation, by φ. Then the scheme J' defined as the fiber product

$$\begin{tikzcd}
X^{(d)} \arrow[d, "(1,\varphi)"'] & J' \arrow[l] \arrow[d] \\
X^{(d)} \times X^{(d)} & X^{(d)} \arrow[l, "\Delta"']
\end{tikzcd} \tag{2.3.1}$$

represents Div_X^d. We will find local sections of this natural transformation, thus obtaining representations of certain subfunctors of P_X^d. We will use this subvarieties to build a representation of P_X^d.

We only have tro prove that P_X^d is representable some $d > 0$. Thus, we can suppose that d is suficiently large, so that for every line bundle $L \in Div_X^d$, $\deg(K \otimes L^{-1}) = \deg(K) - d < 0$. Thus, for every line bundle $L \in Div_X^d$, the linear system $H^0(L)$ of effective divisors D in Div_X^d, such that $f(D) = L$ has dimension $h^0(L) = d + 1 - g$. In order to find a section for f, we reduce the dimension of this system by fixing a family of k-rational points $\gamma = (P_1, \ldots, P_{d-g})$ on X and considering only those effective divisors D such that $D \geq D_\gamma := \sum P_i$. We define the functors

$$X^\gamma(T) = \{D \in \mathrm{Div}_X^r(T) | h^0(D_t - D_\gamma) = 1 \forall t \in T\}$$
$$P_X^\gamma(T) = \{L \in P_X^d(T) | h^0(L_t \otimes L_\gamma^{-1}) = 1 \forall t \in T\}$$

Where L_γ is the line bundle corresponding to D_γ. Proposition [CS86, §7.4.2] and Corollary [CS86, §7.4.3] prove that $X^\gamma(T)$ is representable by open subvarieties

$X^\gamma \subseteq X^{(d)}$, such that $X^{(d)}$ is the union of the subvarieties X^γ. Moreover, P_X^γ is representable by varieties J^γ, that are closed subvarieties of X^γ.

We build the Jacobian $\mathrm{Jac}(X)$ by selecting a covering of $X^{(d)} = \bigcup_{i=1}^m X^{\gamma_i}$ built from a finite set of tuples $\gamma_1, \ldots, \gamma_m$. $\mathrm{Jac}(X)$ is then defined patching together the corresponding varieties J^{γ_i}.

□

Thus, the Jacobian is a fine moduli space for the the functor P_X^0. Its canonical universal family is just a line bundle over $X \times \mathrm{Jac}(X)$ called the Picard bundle.

2.4 Vector bundle moduli space

Once we have analyzed the rank 1 scenario we can study the rank r scenario. Recall that we define a family of vector bundles parametrized by a scheme T to be a vector bundle over $X \times T$. We will provisionally define two families of vector bundles parametrized by a scheme T, say $\mathcal{L}, \mathcal{L}'$, to be equivalent if and only if there exist a line bundle $L \to T$ such that $\mathcal{L}$ and $\mathcal{L}' \otimes \pi^* L$ are isomorphic as vector bundles.

In contrast to the rank 1 case, we will not be able to build even a coarse moduli space for the previously defined moduli problem if we don't impose some additional restrictions to the vector bundles.

This condition will be stated in terms of a ”stability” condition.

2.4.1 Mumford Stability

Definition 2.4.1. *Let E be a vector bundle over X. The slope of E, $\mu(E)$ is*

$$\mu(E) := \frac{\deg(E)}{\mathrm{rk}(E)}$$

The additivity properties of the degree and the rank gives the slope some nice, yet weaker, order property on short exact sequences.

Proposition 2.4.2. *Let A, B, C be vector bundles such that*

$$0 \to A \to B \to C \to 0$$

is a short exact sequence. Then either $\mu(A) \leq \mu(B) \leq \mu(C)$ or $\mu(A) \geq \mu(B) \geq \mu(C)$. Moreover, if some of the inequalities is strict all the inequalities are strict.

In order to prove it we will use the following simple lemma

Lemma 2.4.3. *If $a, b, c, d \in \mathbb{R}$, with $b, d > 0$ and $\frac{a}{b} \leq \frac{c}{d}$ then*

$$\frac{a}{b} \leq \frac{a+c}{b+d} \leq \frac{c}{d}$$

If $\frac{a}{b} < \frac{c}{d}$ then all inequalities are strict.

Proof. As $b, d > 0$, $\frac{a}{b} \leq \frac{c}{d}$ if and only if $ad \leq bc$. Similarly, taking the cross products it is enough to prove that $a(b+d) \leq b(a+c)$ and $(a+c)d \leq (b+d)c$, which are clear, because $ad + ab \leq bc + ab$ and $ad + cd \leq bc + cd$. If the hypothesis is a strict inequality then $ad < bd$ and the proof is analogous. □

Proof of the proposition. Let us first assume that $\mu(A) \leq \mu(C)$. Applying the previous lemma we have that

$$\mu(A) = \frac{\deg(A)}{\mathrm{rk}(A)} \leq \frac{\deg(A) + \deg(C)}{\mathrm{rk}(A) + \mathrm{rk}(C)} \leq \frac{\deg(C)}{\mathrm{rk}\, C} = \mu(C) \tag{2.4.1}$$

As the degree and rank are additive, we have $\deg(A) + \deg(C) = \deg(B)$ and $\mathrm{rk}(A) + \mathrm{rk}(C) = \mathrm{rk}(B)$, so we equation (2.4.1) transforms into

$$\mu(A) \leq \mu(B) \leq \mu(C)$$

If $\mu(A) \geq \mu(C)$ working in an analogous way we obtain $\mu(A) \geq \mu(B) \geq \mu(C)$. Finally, if $\mu(A) \neq \mu(B)$ or $\mu(B) \neq \mu(C)$, then $\mu(A) \neq \mu(C)$ and the lemma proves that the inequalities must be all strict. □

Definition 2.4.4 (Mumford Stabilty)**.** *A vector bundle E is called (semi)stable if for every subbundle $F \neq E$ the following inequality holds*

$$\mu(F)(\leq) < \mu(E)$$

A non-stable semistable vector bundle will be called strictly semistable. A vector bundle that is not semistable is called unstable.

The stability condition for the vector bundle E is only tested for subbundles of E, but we can prove that it is equivalent to test it for all subsheaves $F \hookrightarrow E$.

Proposition 2.4.5. *A vector bundle E is (semi)stable if and only if for every subsheaf $F \hookrightarrow E$,*

$$\mu(F)(\leq) < \mu(E)$$

Proof. As every subbundle is a subsheaf, it is clear that a bundle that fulfills the hypothesis of the proposition is stable (respectively semistable).

Let E be a (semi)stable vector bundle. Let F be a subsheaf of E and let $\overline{F}$ be its saturation. By proposition 1.4.7, $\deg(\overline{F}) \geq \deg(F)$. As $\mathrm{rk}(\overline{F}) = \mathrm{rk}(F)$, we get that $\mu(\overline{F}) \geq \mu(F)$. E is (semi)stable and $\overline{F}$ is a subbundle of E, so

$$\mu(F) \leq \mu(\overline{F})(\leq) < \mu(E)$$

□

Although we can't expect a moduli space that represents all the vector bundles on a curve, we will be able to build a moduli space of semistable bundles modulo a certain equivalence relation later on. The importance of the stability goes beyond this, as we will see that the points of this moduli space corresponding to classes of stable vector bundles will all belong to the smooth locus of the moduli.

As an example, we will check that the stability conditions are compatible with the product with line bundles. This will guaranty that we will be able to define the corresponding equivalence relation for families later.

Proposition 2.4.6. *If E is a (semi)stable vector bundle and L is a line bundle then $E \otimes L$ is (semi)stable.*

Proof. We have $\deg(E \otimes L) = \deg(E) + \operatorname{rk}(E)\deg(L)$ and $\operatorname{rk}(E \otimes L) = \operatorname{rk}(E)$, so

$$\mu(E \otimes L) = \frac{\deg(E \otimes L)}{\operatorname{rk}(E \otimes L)} = \frac{\deg(E) + \operatorname{rk}(E)\deg(L)}{\operatorname{rk}(E)} = \mu(E) + \deg(L)$$

Suppose that F is a proper subbundle of $E \otimes L$ that contradicts the stability condition, i.e., $\mu(F)(>) \geq \mu(E \otimes L) = \mu(E) + \deg(L)$. Then $F \otimes L^{-1}$ is a subbundle of E such that

$$\mu(F \otimes L^{-1}) = \mu(F) - \deg(L)(>) \geq \mu(E)$$

contradicting the hypothesis that E is (semi)stable. $\square$

For a general curve, there may exist stable, strictly semistable and unstable vector bundles. An example of all three kinds of bundles can be built from a line bundle $L \to X$ with nonzero degree d. L itself is clearly stable, as the only proper subbundle is the trivial one.

Let us now take $E = L \oplus L$. We will see that E is a semistable line bundle. Clearly, E is not stable, as $\mu(E) = \frac{\deg(L \oplus L)}{\operatorname{rk}(L \oplus L)} = \frac{2\deg(L)}{2\operatorname{rk}(L)} = \mu(L)$. Suppose that E is unstable. Then there exist a subbundle F such that $\mu(F) > \mu(E)$. Let $i : F \to E$ be the inclusion map, and let $\pi_1, \pi_2 : E = L \oplus L \to L$ be the canonical projections of E into the two copies of L that form $L \oplus L$. Thus, the composition $\pi_1 \circ i$ and $\pi_2 \circ i$ are vector bundle morphisms $F \to L$. As $F \to E$ is nonzero, then at least one of the morphisms $\pi_1 \circ i$ or $\pi_2 \circ i$ must be nonzero. Thus $\operatorname{Hom}(F, L) = H^0(F^{-1} \otimes L)$ is nonzero.

Nevertheless, as $\deg(F) = \mu(F) > \mu(E) = \mu(L) = \deg(L)$, then $\deg(F^{-1} \otimes L) = \deg(L) - \deg(F) < 0$. Thus, $\dim(H^0(F^{-1} \otimes L)) = 0$.

Without loss of generality, we can suppose that $d > 0$. Otherwise we can take L^{-1}. If we take $E = L \oplus L \otimes L$, we have the exact sequence

$$0 \to L \otimes L \to E \to L \to 0$$

so $\deg(E) = \deg(L \otimes L) + \deg(L) = 3\deg(L) = 3d$. Clearly $\operatorname{rk}(E) = 2$, so $\mu(E) = 3d/2$. On the other side, $L \otimes L$ is a proper subbundle of E with degree $2d$ and rank 1, so it has slope $\mu(L \otimes L) = 2d > \mu(E)$. Thus, E is unstable.

The stability conditions impose moreover some restrictions to morphisms between vector bundles.

Proposition 2.4.7. *Let E and F be two semistable vector bundles. If $\mu(E) > \mu(F)$ then* $\operatorname{Hom}(E, F) = 0$.

Proof. Let us suppose that $f : E \to F$ is a nonzero map. Then

$$0 \to \operatorname{Ker}(f) \to E \to \operatorname{Im}(f) \to 0$$

is exact. As E is semistable and $\operatorname{Ker}(f)$ is a subbundle, $\mu(\operatorname{Ker}(f)) \leq \mu(E)$. By proposition 2.4.2, $\mu(E) \leq \mu(\operatorname{Im}(f))$. On the other hand, $\operatorname{Im}(f)$ is a subbundle of F. F is semistable, so $\mu(\operatorname{Im}(f)) \leq \mu(F)$. Combining both inequalities we get $\mu(E) \leq \mu(F)$. $\square$

Proposition 2.4.8. *Let E and F be two stable vector bundles. If $\mu(E) = \mu(F)$ then every nonzero map is an isomorphism.*

Proof. Let $f : E \to F$ be a nonzero morphism. E is stable, so either $\ker(f) = 0$ or $\mu(\mathrm{Ker}(f)) < \mu(E)$. By proposition 2.4.2, $\mu(E) < \mu(\mathrm{Im}(f)) \leq \mu(F)$, which is impossible, so $\ker(f) = 0$. F is stable, then either $\mathrm{Im}(f) = F$ or $\mu(E) \leq \mu(\mathrm{Im}(f)) < \mu(F)$. The later is not possible, so f is an isomorphism. □

The following proposition allows us to decompose any semistable vector bundle in terms of a filtration with a grading of stable vector bundles.

Proposition 2.4.9 (Jordan-Hölder filtration)**.** *Let E be a semistable vector bundle. There exists a filtration*

$$0 = E_0 \subset E_1 \subset \cdots \subset E_{n-1} \subset E_n = E$$

with E_i/E_{i-1} stable and $\mu(E_i/E_{i-1}) = \mu(E)$.

Proof. First of all, if E is stable then it is enough to take the trivial filtration $0 \subset E$. Otherwise, we proceed by induction on the rank of E. As all line bundles are clearly stable, the result holds for vector bundles of rank one. Let us assume that the result works for all semisteble vector bundles of rank less than $\mathrm{rk}(E)$. Without loss of generality we can assume that E is strictly semistable. Let us take E_{n-1} as a subbundle of E maximal among those of maximum slope. As E is strictly semistable, $\mu(E_{n-1}) = \mu(E)$.

Every subbundle F of E_{n-1} is a subbundle of E, so by maximality of $\mu(E_{n-1})$ among the subbundles of E, $\mu(F) \leq \mu(E_{n-1})$. Therefore, E_{n-1} is semistable. We have the following exact sequence

$$0 \to E_{n-1} \to E \to E/E_{n-1} \to 0$$

So $\deg(E_{n-1}) + \deg(E/E_{n-1}) = \deg(E)$ and $\mathrm{rk}(E_{n-1}) + \mathrm{rk}(E/E_{n-1}) = \mathrm{rk}(E)$. As $\mu(E) = \mu(E_{n-1})$, then $\mu(E) = \mu(E_{n-1}) = \mu(E/E_{n-1})$.

Let us prove that E/E_{n-1} is stable. There is a one to one correspondence between subbundles of E/E_{n-1} and subbundles F of E such that $E_{n-1} \subseteq F \subseteq E$. Let F/E_{n-1} be a proper subbundle of E/E_{n-1} that contradicts the stability condition. The following exact sequence holds

$$0 \to F/E_{n-1} \to E/E_{n-1} \to E/F \to 0$$

By hypothesis, $\mu(F/E_{n-1}) \geq \mu(E/E_{n-1})$. Then, proposition 2.4.2 implies that $\mu(E/E_{n-1}) \geq \mu(E/F)$. On the other hand, we have the following exact sequence

$$0 \to F \to E \to E/F \to 0$$

We have proved that $\mu(E) = \mu(E/E_{n-1}) \geq \mu(E/F)$, again by proposition 2.4.2 we get that $\mu(F) \geq \mu(E)$. As E is semistable we must have $\mu(F) = \mu(E)$, but E_{n-1} was chosen as the one with maximal fulfilling this property, so E_{n-1} can't be a subbundle of F unless $E = F$. Clearly, $E/F = 0$ does not contradict the stability condition for E/E_{n-1}, so E/E_{n-1} is stable.

Finally, the rest of the filtration is built from E_{n-1} by induction hypothesis. □

A filtration as such of proposition 2.4.9 will be called a Jordan-Hölder filtration for the vector bundle E. n is the length of the filtration and we will call $Gr(E) := \bigoplus(E_i/E_{i-1})$ the grading of E. The following theorem proves that the grading $Gr(E)$ is unique up to isomorphism.

Theorem 2.4.10 (Jordan-Hölder's Theorem)**.** *Let E be a vector bundle. All the Jordan-Hölder filtrations for E have the same length. The grading $Gr(E)$ is unique up to isomorphism in the sense that if (E_i) and (E'_i) are two filtrations, there exist a permutation $\sigma \in \Sigma_n$ such that*

$$E_i/E_{i-1} \cong E'_{\sigma(i)}/E'_{\sigma(i-1)}$$

Proof. We will act by induction on the rank of E. If E is a line bundle, the unique possible filtration is the trivial one. Suppose that the theorem is true for vector bundles of rank less than $\mathrm{rk}(E)$. Let (E_i) and (E'_i) two Jordan-Hölder filtrations for E of lengths n and n' respectively. Let j be the integer such that $E_1 \subseteq E'_j$ and $E_1 \nsubseteq E'_{j-1}$. There exist such j because $E_1 \subset E'_n = E$, and $E_1 \nsubseteq E'_0 = 0$. Thus we have a canonical map $E_1 \to E'_j/E'_{j-1}$ that is nonzero. By proposition 2.4.8, $E_1 = E_1/E_0 \to E'_j/E'_{j-1}$ is an isomorphism. Hence, there is an isomorphism $E'_j \cong E'_{j-1} \oplus E_1$, so there is a short exact sequence

$$0 \to E'_{j-1} \to E/E_1 \to E/E'_j \to 0$$

Let us consider the vector bundle E/E_1. The filtration (E_i) induces trivially a Jordan-Hölder filtration $(E_i/E_1)_{i=2}^n$ on E/E_1. The previous short exact sequence implies that (E'_i) also induces a second filtration $(E''_i)_{i=1}^{n'-1}$ on E/E_1 taking $E''_i = E'_i$ for $i < j$ and E''_i as the preimage of E'_{i+1}/E'_j by the map $E/E_1 \to E/E'_j$. By induction hypothesis applied to E/E_1 and filtrations $(E_i/E_1)_{i=2}^n$ and $(E''_i)_{i=1}^{n'-1}$ we get $n = n'$ and

$$\bigoplus_{i=2}^{n} E_i/E_{i-1} = \bigoplus_{i=2}^{n}(E_i/E_1)/(E_{i-1}/E_1) \cong \bigoplus_{i=1}^{n'-1} E''_i/E''_{i-1}$$

For $i \in \{1, \dots, j-1\}$, we have $E''_i/E''_{i-1} = E'_i/E'_{i-1}$. For $i = j$ we have $E''_j/E''_{j-1} \cong E'_{j+1}/(E'_{j-1} \oplus E_1) \cong E'_{j+1}/E'_j$. If $i > j$ we get simply $E''_j/E''_{j-1} \cong E'_{j+1}/E'_j$. Therefore

$$\bigoplus_{i=2}^{n} E_i/E_{i-1} =\cong \bigoplus_{i=1}^{n'-1} E''_i/E''_{i-1} \cong \bigoplus_{i \neq j} E'_i/E'_{i-1}$$

As $E_1/E_0 \cong E_j/E_{j-1}$ we obtain the desired isomorphism of graded objects. □

Finally we can study the structure of a general vector bundle building a similar filtration based on semistable vector bundles.

Theorem 2.4.11 (Harder-Narasimhan filtration)**.** *Let E be an algebraic vector bundle. Then E has an increasing filtration by vector sub-bundles*

$$0 \subset E_1 \subset E_2 \subset \cdots \subset E_n = E$$

where E_i/E_{i-1} is semi-stable and $\mu(E_i/E_{i-1}) > \mu(E_{i+1}/E_i)$. Moreover the filtration is unique.

Similarly to the Jordan-Hölder filtration, the Harder-Narasimhan filtration is built inductively, taking E_{n-1} as a subsheaf of E of maximal slope and maximal rank amongst the subsheaves of maximal slope. As the quotient of a vector bundle by a subbundle is a vector bundle, we can repeat the construction with the quotient. A complete proof of this theorem can be found in [Por97, Proposition 5.4.2].

Using the Jordan-Hölder filtration we can define an equivalence class for semistable vector bundles.

Definition 2.4.12. *Two semistable bundles E and E' are said to be S-equivalent if $Gr(E) \cong Gr(E')$.*

By construction, if E is a stable vector bundle, its Jordan-Hölder filtration is the trivial one, $0 \subset E$. Thus, if E or E' are stable, they are S-equivalent if and only if thy are isomorphic. The S-equivalence is crucial in the definition of the moduli space of vector bundles.

2.4.2 Moduli problem

Let X be an algebraic curve of genus g. We will consider the set $\mathcal{A}$ of semistable vector bundles over X of rank r and degree d, together with the equivalence relation given by the S-equivalence. Let us consider a family of elements of $\mathcal{A}$ parametrized by a scheme T as a vector bundle over $X \otimes T$. We will say that two families $E \to X \otimes T$ and $E' \to X \otimes T$ parametrized by T are equivalent if one of the following conditions hold.

1. If $T = \{pt\}$, $E \sim E'$ if they are S-equivalent.
2. Otherwise, $E \sim E'$ if there exist a line bundle $L \to T$ such that $E' \cong E \otimes \pi^* L$, where $\pi : X \times T \to T$ is the canonical projection.

The pullback of a family of vector bundles will be given by the pullback of sheaves by a scheme morphism. It is easy to check that the equivalence relation of families is compatible with the pullback. Therefore, the previous notions form a well defined moduli problem.

Narasimhan and Ramanan [NR69] proved that there exist a coarse moduli space solving the previous moduli problem. We will denote the corresponding scheme by $\mathcal{M}(r, d, X)$. In order to simplify the notation, we will omit the curve whenever it is clear from the context. It is a normal quasi-projective variety of dimension

$$\dim(\mathcal{M}(r, d)) = r^2(g - 1) + 1$$

Tyurin [Tyu70] proved that if r and d are coprime, then the moduli space of S-equivalence classes of semistable vector bundles over X of rank r and degree d is fine. Later on, Ramanan [Ram73] proved the converse, i.e., that if $\mathcal{M}(r, d)$ is fine, then r and d must be coprime.

If we restrict the problem to the set of stable vector bundles, Mumford [Mum62] proved using geometric invariant theory [Mum82] that there exist a coarse moduli space parameterizing the subproblem and Seshadri [Ses67] constructed its compactification using the notion of S-equivalence defined in [NS65]. We will denote this scheme by $\mathcal{M}^s(r,d)$. It is a normal quasi-projective open subvariety of $\mathcal{M}(r,d)$. Seshadri [Ses67] also proved that this variety was non-singular, so $\mathcal{M}^s(r,d)$ lies inside the smooth locus of $\mathcal{M}(r,d)$. Furthermore, Narasimhan and Ramanan proved the following theorem [NR69, Theorem 1].

Theorem 2.4.13. *Let X be a non-singular irreducible complete algebraic curve of genus $g \geq 2$. Let $r \geq 2$. Then the set of singular points of the variety $\mathcal{M}(r,d)$ is precisely the set of non-stable points, except when $g = 2$, $r = 2$ and d is even.*

As an example of an exceptional case, [NR69, §7 Theorem 1] proved that $\mathcal{M}(2,0)$ is smooth.

The previous moduli problem is defined for $\mathrm{GL}(r,\mathbb{C})$-bundles, but later on we will see that the subvarieties of $\mathcal{M}(r,d)$ corresponding to $\mathrm{SL}(r,\mathbb{C})$-vector bundles with a prescribed determinant is of great interest.

Let ξ be a line bundle of degree d over X. We denote by $\mathcal{M}(r,\xi) = \mathcal{M}(r,\xi,X)$ the moduli space of S-equivalence classes of semistable vector bundles E over X of rank r, together with an isomorphism $\det(E) \cong \xi$. As the condition $\det(E) \cong \xi$ is closed, $\mathcal{M}(r,\xi)$ is a closed subspace of $\mathcal{M}(r,d)$. It can be proved that it is a coarse moduli space for all ξ, and that it is a fine moduli space if and only if $\deg(\xi)$ is coprime with r.

$\mathcal{M}(r,\xi)$ is a normal quasi-projective subvariety of $\mathcal{M}(r,d)$ of dimension

$$\dim(\mathcal{M}(r,\xi)) = (r^2-1)(g-1)$$

Similarly, we will denote the open subvariety of $\mathcal{M}(x,\xi)$ corresponding to stable bundles by $\mathcal{M}^s(r,\xi)$. As $\mathcal{M}^s(r,d)$ is smooth, $\mathcal{M}^s(r,\xi)$ is a smooth quasi-projective variety.

Chapter 3

Higgs bundles

The geometry of Higgs bundles has been a mainstream topic in algebraic geometry during the last decades. Higgs bundles where first introduced by Hitchin [Hit87a] while working on self-duality equations. The similarities of the equations with the physical model describing the Higgs boson leaded up to the name "Higgs field". Later on, the geometry of Higgs bundles were studied by Simpson [Sim92].

In this chapter we will build the moduli space of Higgs bundles and we will state some of its geometric properties. Particularly, we will introduce the Hitchin map and analyze its algebro-geometric characteristics. Let X be a Riemann surface of genus $g \geq 2$ and let K be the canonical bundle over X.

Definition 3.0.1. *A Higgs bundle over the Riemann surface X is a pair (E, Φ), where E is a vector bundle over X and Φ is a sheaf homomorphism $\Phi : E \to E \otimes K$ called Higgs field.*

In order to describe the category of Higgs bundles completely, we must define the corresponding morphisms.

Definition 3.0.2. *A morphism of Higgs bundles $f : (E, \Phi) \to (E', \Phi')$ is a vector bundle morphism $f : E \to E'$ such that the induced morphism $\tilde{f} : E \otimes K \to E' \otimes K$ make following diagram commutative.*

$$\begin{tikzcd} E \otimes K \arrow[r, "\tilde{f}"] & E' \otimes K \\ E \arrow[u, "\Phi"] \arrow[r, "f"] & E' \arrow[u, "\Phi'"'] \end{tikzcd} \tag{3.0.1}$$

Using sheaf theory, we can define alternatively the Higgs field over a vector bundle $E \to X$ as an element of $H^0(X, K \otimes \mathrm{End}(E))$. This interpretation will be useful later on, as it describes the set of possible Higgs bundles over a certain vector bundle in cohomological terms.

3.1 Stability conditions for Higgs bundles

Similarly to vector bundles, we can't expect to have a moduli space of Higgs bundles. We need to define a suitable stability condition and an equivalence relation for families of Higgs bundles. We will use a version of Mumford stability. In this case, we will not check the slope condition for all subbundles, but only for those preserved by the Higgs field in the following sense.

Definition 3.1.1. *Let (E, Φ) be a Higgs bundle. A subsheaf F of E is said to be Φ-invariant if*

$$\Phi(F) \subseteq F \otimes K$$

Therefore, we define Mumford stability for Higgs bundles as follows.

Definition 3.1.2. *A parabolic Higgs bundle (E, Φ) is called (semi)stable whenever for all Φ-invariant subbundles $F \subsetneq E$*

$$\mu(F)(\leq) < \mu(E)$$

As it happened in the case of vector bundles, we are defining the stability condition to be tested only for subbundles and not for subsheafs. In the previous chapter we discussed how this was equivalent to testing all subsheafs, as the saturation of each subsheaf of a vector bundle is a subbundle with higher slope. Now, there exist an extra condition for a subbundle to be tested. Only Φ-invariant subbundles are being considered. The following lemma proves that this is again equivalent as testing Φ-invariant subsheafs.

Lemma 3.1.3. *Let (E, Φ) be a Higgs bundle. Let E' be a Φ-invariant subsheaf of E. Then the saturation of E', $\overline{E'}$ is a Φ-invariant subbundle of (E, Φ).*

Proof. By construction of the saturation, we obtain exact sequences

$$0 \longrightarrow \overline{E'} \longrightarrow E \xrightarrow{p} C \longrightarrow 0 \tag{3.1.1}$$

$$0 \longrightarrow \overline{E'} \otimes K \longrightarrow E \otimes K \xrightarrow{p} C \otimes K \longrightarrow 0$$

where C is the quotient of E/E' with its torsion sheaf T, and thus, is a torsion free sheaf.

Let us suppose that there exist an open set U and a section $s \in \overline{E'}(U)$ such that $\Phi(s) \notin (\overline{E'} \otimes K)(U)$. We proved that there is an exact sequence

$$0 \longrightarrow E' \longrightarrow \overline{E'} \xrightarrow{q} T \longrightarrow 0 \tag{3.1.2}$$

If $s \in E'(U)$ then, as E' is Φ-invariant, we would have $s \in (E' \otimes K)(U) \subseteq (\overline{E'} \otimes K)(U)$. Thus we can suppose that $s \notin E'(U)$. As the sequence (3.1.2) is exact, $E' = \mathrm{Ker}(q)$, so $q(s) \neq 0$. T is a torsion sheaf, so $q(s)$ is a torsion element and there exist $f \in \mathcal{O}_X(U)$ such that $fq(s) = 0$. As q is a morphism of $\mathcal{O}_X$-sheaves, $q(fs) = 0$ and therefore $fs \in E'$.

Then, we know that $\Phi(fs) \in (E' \otimes K)(U)$. As Φ is also a morphism of $\mathcal{O}_X$-modules, we get that $f\Phi(s) \in (E' \otimes K)(U)$.

In this way, we have found an element $\Phi(s) \in (E \otimes K)(U)$ such that $\Phi(s) \notin (\overline{E'} \otimes K)$ but $f\Phi(s) \in (E' \otimes K)$. This is impossible, as $\overline{E'} \otimes K$ is torsion free. $\square$

An alternative way of understanding the previous lemma is through the following commutative diagram.

$$
\begin{tikzcd}
 & 0 \arrow[rr] & & E'\otimes K \arrow[rr] \arrow[dd, hook] & & E\otimes K \arrow[rr] \arrow[dd, equal] & & E/E'\otimes K \arrow[rr] \arrow[dd] & & 0 \\
0 \arrow[rr] & & E' \arrow[rr] \arrow[ur, "\Phi"] \arrow[dd, hook] & & E \arrow[rr] \arrow[ur, "\Phi"] \arrow[dd, equal] & & E/E' \arrow[rr] \arrow[ur, "\Phi"] \arrow[dd, two heads] & & 0 & \\
 & 0 \arrow[rr] & & \overline{E'}\otimes K \arrow[rr] & & E\otimes K \arrow[rr] & & C\otimes K \arrow[rr] & & 0 \\
0 \arrow[rr] & & \overline{E'} \arrow[rr] \arrow[ur, "\Phi"] & & E \arrow[rr] \arrow[ur, "\Phi"] & & C \arrow[rr] \arrow[ur, "\Phi"] & & 0 &
\end{tikzcd}
\tag{3.1.3}
$$

E' begin Φ-invariant makes the diagram

$$
\begin{tikzcd}
0 \arrow[r] & E'\otimes K \arrow[r] & E\otimes K & & \\
0 \arrow[r] & E' \arrow[r] \arrow[u, "\Phi"] & E \arrow[r] \arrow[u, "\Phi"] & E/E' \arrow[r] & 0
\end{tikzcd}
\tag{3.1.4}
$$

commutative. Taking the quotient in the upper line it is proved that, regarding E/E' as a subsheaf of E, it is Φ-invariant. Let T be the torsion sheaf of E/E'. For any open set U, let $t \in T(U)$. It is clear that $t \otimes k$ is a torsion element of $E/E' \otimes K$ for any $k \in K(U)$. Therefore, $T \otimes K$ lies in the torsion sheaf of $E/E' \otimes K$. The sheaf K being torsion free implies that there can't be any other torsion elements, so $T \otimes K$ is the torsion sheaf of $E/E' \otimes K$.

As Φ is a $\mathcal{O}_X$-module morphism, the image of a torsion element under Φ is a torsion element, so the diagram

$$
\begin{tikzcd}
0 \arrow[r] & T\otimes K \arrow[r] & E/E'\otimes K & & \\
0 \arrow[r] & T \arrow[r] \arrow[u, "\Phi"] & E/E' \arrow[r] \arrow[u, "\Phi"] & C \arrow[r] & 0
\end{tikzcd}
\tag{3.1.5}
$$

commutes. Again, taking the quotient proves that C is Φ-invariant, so the diagram

$$
\begin{tikzcd}
 & & E\otimes K \arrow[r] & C\otimes K \arrow[r] & 0 \\
0 \arrow[r] & \overline{E'} \arrow[r] & E \arrow[r, "p"] \arrow[u, "\Phi"] & C \arrow[r] \arrow[u, "\Phi"] & 0
\end{tikzcd}
\tag{3.1.6}
$$

commutes. Thus, p is a morphism of Higgs bundles with Higgs field Φ, so its kernel $\overline{E'}$ is a Φ invariant subbundle of E. Completing diagrams (3.1.4), (3.1.5) and (3.1.6) in the described way and putting them all together we obtain that the diagram (3.1.3) is commutative.

On the other hand, the stability condition of Higgs bundles is not equivalent to the stability condition for its underlying vector bundle. Obviously, if the underlying vector bundle is (semi)stable, then the Higgs field is (semi)stable, but the reciprocal is not true. A stable Higgs bundle may have an unstable vector bundle.

Nevertheless, some of the main theorems about (semi)stable bundles described in chapter 2 still hold for Higgs bundles. In particular, there exist Jordan-Hölder and Harder-Narashimhan filtrations for Higgs bundles.

Theorem 3.1.4 (Jordan-Hölder filtration for Higgs bundles)**.** *Let (E,Φ) be a semistable Higgs bundle. There exists a filtration of Φ-invariant subbundles of E,*

$$0 = E_0 \subset E_1 \subset \cdots \subset E_{n-1} \subset E_n = E$$

such that $(E_i/E_{i-1}, \Phi|_{E_i/E_{i-1}})$ is a stable Higgs bundle and $\mu(E_i/E_{i-1}) = \mu(E)$. Moreover, all the Jordan-Hölder filtrations for E have the same length and the grading

$$Gr(E,\Phi) = \bigoplus_i (E_i/E_{i-1}, \Phi|_{E_i/E_{i-1}})$$

is unique up to isomorphism.

Theorem 3.1.5 (Harder-Narasimhan filtration for Higgs bundles)**.** *Let (E,Φ) be a Higgs bundle. Then E has an increasing filtration by vector Φ-invariant subbundles*

$$0 \subset E_1 \subset E_2 \subset \cdots \subset E_n = E$$

where $(E_i/E_{i-1}, \Phi|_{E_i/E_{i-1}})$ is a semistable Higgs bundle and $\mu(E_i/E_{i-1}) > \mu(E_{i+1}/E_i)$. Moreover the filtration is unique.

Therefore, we can define two semistable Higgs bundles to be S-equivalent if the gradings of their corresponding Jordan-Hölder filtrations are isomorphic.

3.2 Moduli space of Higgs bundles

The coarse moduli space of Higgs bundles was constructed by Hitchin [Hit87b]. It parametrizes S-equivalence classes of semistable Higgs bundles of rank r and degree d over a Riemann surface X.

We define a family of semistable Higgs bundles over a Riemann surface X parametrized by a scheme T to be a vector bundle E over $X \times T$ together with an element $\Phi \in \Gamma(T, (\pi_T)_*(\pi_X^* K \otimes \mathrm{End}(E)))$, where $\pi_X : X \times T \to X$ and $\pi_T : X \times T \to T$ are the canonical projections. Notice that for every $t \in T$, $\Phi(t)$ is an element of $K \otimes \mathrm{End}(E_t)$. Thus, it is a Higgs fileld over E_t. Using the S-equivalence relation, we will define two families of Higgs bundles parametrized by T to be equivalent if the following conditions hold.

a) If $T = \{pt\}$, then two families are equivalent if the corresponding Higgs bundles are S-equivalent.

b) Otherwise, two families are equivalent if there exist a line bundle $L \to T$ such that
$$E \cong E \otimes \pi^* L$$
where $\pi : X \times T \to T$ is the canonical projection.

We will denote this moduli space by $\mathcal{M}_{Higgs}(r,d,X)$. We will also consider the subvariety of this moduli corresponding to Higgs bundles (E,Φ) whose underlying vector bundle E is a $\mathrm{SL}(r,\mathbb{C})$-vector bundle with prescribed determinant $\deg(E)\cong\xi$ of degree d and such that $\mathrm{tr}(\Phi)=0$. We will denote this space by $\mathcal{M}_{Higgs}(r,\xi,X)$. We will drop the curve from the notation whenever it is clear from the context.

Remark 3.2.1. *The condition* $\mathrm{tr}(\Phi)=0$ *is derived from the general construction of Higgs bundles on G-principal bundles.*

Definition 3.2.2. *Given a sheaf of groups G, let $\mathcal{G}=\mathrm{Lie}(G)$. A G-Higgs bundle is a pair (E,Φ), where E is a principal G-bundle and $\Phi\in H^0(E(\mathcal{G})\otimes K)$. $E(\mathcal{G})$ is defined as the G-bundle whose fibers are those of $\mathcal{G}$ and whose transitions functions are those of E. Alternatively, this bundle can be described as*

$$E(\mathcal{G}):=E\times\mathcal{G}/G$$

The definition of $E(\mathcal{G})$ is possible thanks to the fact that G acts both on E and $\mathcal{G}$. G acts on E because of the structure of G-principal bundle, and it acts on $\mathcal{G}$ by the adjoint action. Thus, we can get the quotient of $E\times\mathcal{G}$ by G.

If we take $G=\mathrm{SL}(\mathcal{O}_X^{\oplus(r-1)}\oplus\xi)$, $E(\mathcal{G})$ corresponds to traceless endomorphisms of E. Therefore, $H^0(E(\mathcal{G})\otimes K)$ is precisely the set of traceless homomorphisms $E\to E\otimes K$, i.e., traceless Higgs fields.

We will also denote by $\mathcal{M}^s_{Higgs}(r,d)$ and $\mathcal{M}^s_{Higgs}(r,\xi)$ the subspaces corresponding to stable Higgs bundles with the corresponding rank and a fixed degree or determinant respectively. Similarly to the moduli of vector bundles, it can be proved that they are open subvarieties that lie in the smooth locus of the corresponding moduli spaces, so they are smooth.

Both spaces are irreducible varieties. Moreover, if $r=2$, they are connected [Nit91, Theorem 7.5]. If r and d are coprime then we can prove that every semistable Higgs bundle is a stable Higgs bundle. Let (E,Φ) be a semistable Higgs bundle of rank r and degree d, with r and d coprime. Let $F\subsetneq E$ be a Φ invariant subbundle. Suppose that $\mu(F)=\mu(E)$. Then

$$\deg(F)r=\deg(F)\,\mathrm{rk}(E)=\deg(E)\,\mathrm{rk}(F)=d\,\mathrm{rk}(F)$$

Thus, r divides $d\,\mathrm{rk}(F)$. As r and d are coprime we must have $r|\,\mathrm{rk}(F)$, but this is impossible, because $\mathrm{rk}(F)<r$. Therefore, $\mu(F)<\mu(E)$.

Then, if r and d are coprime

$$\mathcal{M}_{Higgs}(r,d)=\mathcal{M}^s_{Higgs}(r,d)\qquad\mathcal{M}_{Higgs}(r,\xi)=\mathcal{M}^s_{Higgs}(r,\xi)$$

and $\mathcal{M}_{Higgs}(r,d)$ and $\mathcal{M}_{Higgs}(r,\xi)$ are smooth.

The moduli space of (semi)stable vector bundles $\mathcal{M}(r,d)$ can be embedded into $\mathcal{M}_{Higgs}(r,d)$ by taking the Higgs field as the zero morphism. All the subbundles of a Higgs bundle with trivial Higgs field are Φ-invariant, so such a Higgs bundle is (semi)stable whenever its underlying vector bundle is (semi)stable. Thus, we get a natural embedding

$$\mathcal{M}(r,d)\hookrightarrow\mathcal{M}_{Higgs}(r,d)$$

$$E \mapsto (E, 0)$$

Fixing the determinant, in both sides, this induces an embedding

$$i : \mathcal{M}(r,\xi) \hookrightarrow \mathcal{M}_{Higgs}(r,\xi) \tag{3.2.1}$$

defined by $E \mapsto (E, 0)$. Let $\mathcal{M}^{st}_{Higgs}(r,\xi)$ be the locus of Higgs bundles (E,Φ) whose underlying vector bundle E is stable. It is an open dense subset of $\mathcal{M}_{Higgs}(r,\xi)$.

From now on, we will focus on the moduli space of Higgs bundles with a prescribed determinant ξ and traceless Higgs field. If X has genus $g \geq 2$, the moduli $\mathcal{M}_{Higgs}(r,\xi)$ is a normal quasiprojective variety of dimension

$$\dim(\mathcal{M}_{Higgs}(r,\xi)) = 2(r^2-1)(g-1)$$

Let

$$\mathrm{pr}_E : \mathcal{M}^{st}_{Higgs}(r,\xi) \longrightarrow \mathcal{M}^s(r,\xi) \tag{3.2.2}$$

be the forgetful map defined by $(E,\Phi) \to E$. By deformation theory, the tangent space at $[E]$, $T_{[E]}\mathcal{M}^s(r,\xi)$ is isomorphic to $H^1(X, \mathrm{End}(E))$. By Serre duality,

$$H^1(X, \mathrm{End}(E))^* \cong H^0(X, \mathrm{End}(E) \otimes K)$$

and hence, the Higgs field is an element of the cotangent bundle $T^*_{[E]}\mathcal{M}^s(r,\xi)$ and one has a canonical isomorphism

$$\mathcal{M}^{st}_{Higgs}(r,\xi) \xrightarrow{\sim} T^*\mathcal{M}^s(r,\xi) \tag{3.2.3}$$

of varieties over $\mathcal{M}^s(r,\xi)$.

3.3 Hitchin map

We will now introduce the definition of the Hitchin map and the Hitchin space. Let $S = \mathbb{V}(K)$ be the total space of the line bundle K, let

$$p : S = \underline{\mathrm{Spec}}\,\mathrm{Sym}^\bullet(K^{-1}) \longrightarrow X$$

be the projection, and $x \in H^0(S, p^*(K))$ be the tautological section. The characteristic polynomial of a Higgs field

$$\det(x \cdot \mathrm{id} - p^*\Phi) = x^r + \tilde{s_1}x^{r-1} + \tilde{s_2}x^{r-2} + \cdots + \tilde{s_r}$$

defines sections $s_i \in H^0(X, K^i)$, such that $\tilde{s_i} = p^*s_i$ and K^i denotes the tensor product of i copies of K (i-th power of K). We define the Hitchin space as

$$\mathcal{H} = \bigoplus_{i=1}^{r} H^0(K^i) \tag{3.3.1}$$

The Hitchin map is defined as

$$H : \mathcal{M}_{Higgs}(r,d) \longrightarrow \mathcal{H} \tag{3.3.2}$$

sending each Higgs bundle (E, Φ) to the characteristic polynomial of Φ.

We can now restrict the Hitchin map to $\mathcal{M}_{Higgs}(r, \xi)$. In order to fix the determinant, we are asking Φ to be traceless, so $s_1 = 0$ and the image in the Hitchin space lies in

$$\mathcal{H}_0 = \bigoplus_{i=2}^{r} H^0(K^i) \tag{3.3.3}$$

We will call $\mathcal{H}_0$ the traceless Hitchin space. Therefore, one obtains a map

$$H : \mathcal{M}_{Higgs}(r, \xi) \longrightarrow \mathcal{H}_0 \tag{3.3.4}$$

We can compute the dimensions of $\mathcal{H}$ and $\mathcal{H}_0$ using the Riemann-Roch theorem and the Serre duality. The dimension of $\mathcal{H}$ is given by

$$\dim(\mathcal{H}_0) = \sum_{i=2}^{r} \dim(H^0(k^i)) \tag{3.3.5}$$

Applying Serre duality, the Riemann-Roch theorem yields,

$$\dim(H^0(K^i)) - \dim(H^0(K^{1-i})) = \deg(K^i) - g + 1$$

For $i \geq 2$, $\deg(K^i) = i(2g-2)$, so $\deg(K^{1-i}) < 0$. Thus $\dim(H^0(K^{1-i})) = 0$ and we obtain

$$\dim(H^0(K^i)) = i(2g-2) - g + 1 = (2i-1)(g-1)$$

If we substitute this dimension in equation (3.3.5), we get

$$\dim(\mathcal{H}_0) = \sum_{i=2}^{r} (2i-1)(g-1) = (r^2-1)(g-1) = \dim(\mathcal{M}(r, \xi))$$

On the other hand, we have $\mathcal{H} = \mathcal{H}_0 \oplus H^0(K)$. As $\dim(H^0(K)) = g$, we obtain that

$$\dim(\mathcal{H}) = \dim(\mathcal{H}_0) + \dim(H^0(K)) = (r^2-1)(g-1) + g = r^2(g-1) + 1 = \dim(\mathcal{M}(r, d))$$

On the other hand, given an element $(s) = (s_1, \ldots, s_r) \in \mathcal{H}$, we define the spectral curve X_s in S as the zero scheme of the following section of p^*K^r

$$f = x^n + \tilde{s_1} x^{n-1} + \tilde{s_2} x^{n-2} + \cdots + \tilde{s_n}$$

where, again, $\tilde{s_i} = p^* s_i$ and $x \in H^0(S, p^*(K))$ is the tautological section. Let π be the restriction of p to X_s. Let us call $\mathcal{I}$ the ideal sheaf generated by the image of the sheaf homomorphism

$$K^{-r} \to \mathrm{Sym}^\bullet(K^{-1})$$

given by $\alpha \mapsto \alpha \sum_{i=0}^{r} s_i$, where we take $s_0 = 1$. We have

$$\pi : X_s = \underline{\mathrm{Spec}}\left(\mathrm{Sym}^\bullet(K^{-1})/\mathcal{I}\right) \longrightarrow X$$

Thus, we obtain the following isomorphism

$$\pi_* \mathcal{O}_{X_s} = \mathcal{O}_X \oplus K^{-1} \oplus K^{-2} \oplus \cdots \oplus K^{-(r-1)}$$

We will now state some of the main properties of the Hitchin map.

Proposition 3.3.1. *The map Hitchin map $H : \mathcal{M}_{Higgs}(r,d) \longrightarrow \mathcal{H}$ is proper.*

The proof can be found in [Nit91, Theorem 6.1]. Then, we would like to describe the generic fibre of the Hitchin map. The next proposition [Hit87b, page 99] proves that the spectral curve is generically smooth.

Proposition 3.3.2. *There is an open dense set U in $\mathcal{H}$ (respectively, $U_0 \in \mathcal{H}_0$) such that the spectral curve X_s is smooth for every $s \in U$ (respectively, in U_0).*

X_s being smooth has some important implications about the structure of the fibre over s, as it can be deduced from the following proposition [Hit87b], [BNR89].

Proposition 3.3.3. *Let $s \in \mathcal{H}$. If X_s is smooth then the fibre $H^{-1}(s)$ is isomorphic to the Jacobian of the spectral curve* $\mathrm{Jac}(X_s)$.

Similarly, if we work on $\mathcal{M}_{Higgs}(r,\xi)$, for every $s \in \mathcal{H}_0$, such that X_s is smooth, the fibre $H^{-1}(s)$ is isomorphic to the Prym variety

$$P_s = \{L \in \mathrm{Pic}(X_s) : \det(\pi_* L) \cong \xi\}$$

As the Jacobian and the Prym varieties are connected, the previous propositions imply that the generic fibre of the Hitchin map is connected. Moreover, all the fibres are equidimensional projective schemes of dimension $(r^2-1)(g-1)$.

The multiplicative group $\mathbb{C}^*$ acts on the moduli space $\mathcal{M}_{Higgs}(r,\xi)$ by

$$t \cdot (E, \Phi) = (E, t\Phi) \tag{3.3.6}$$

The Hitchin map H induces an associated action in $\mathcal{H}_0$ given by

$$t \cdot (v_2, \dots, v_i, \dots, v_r) = (t^2 v_2, \dots, t^i v_i, \dots, t^r v_r) \tag{3.3.7}$$

Where $v_i \in H^0(X, K^i)$ for $i \in 2, \dots, r$. Taking into account this induced action, it is clear that the fixed points of the action (3.3.6) must lie in $0 \in \mathcal{H}_0$. The preimage $H^{-1}(0)$ is called the nilpotent cone. Later on, we will see that the geometry of this subspace of $\mathcal{M}_{Higgs}(r,\xi)$ is specially rich, and it will be of great importance for future theorems. In particular, the canonical immersion of $\mathcal{M}(r,\xi)$ into $\mathcal{M}_{Higgs}(r,\xi)$ lies in the nilpotent cone. It is a reducible scheme. If r and d are coprime, it is a Lagrangian scheme [Lau88], so its irreducible components have all dimension $(r^2-1)(g-1)$.

Chapter 4

Parabolic Vector Bundles

In the previous chapters we have studied moduli problems in which we consider geometric objects built over smooth compact projective curves. Our next objective is to build analogous moduli spaces allowing certain kind of singularities. Let X be a smooth compact curve and let $D = \{x_1, \ldots, x_n\}$ be a set of points over X. Let $U = X \backslash D$ be the punctured curve obtained dropping the points in D from X.

We will consider geometric structures such as Higgs bundles and connections built over the punctured smooth projective curve U such that the singularities at the puncture points will be logarithmic. Thus, we will get some structure defined over a vector bundle over a punctured curve. One way of treating this kind of objects would be to specify a Galois covering of the curve that allows us to eliminate the singularities at the punctures and then control the interaction of the structures with the covering. In this chapter we will take an alternative approach. We will treat structures such as Higgs bundles as defined over a vector bundle on the whole smooth curve and then the singularities at the punctures will be controlled through flags at the points.

4.1 Parabolic bundles and morphisms

Definition 4.1.1. *Let V be a complex vector space. A flag is a decreasing filtration of finite-dimensional subspaces of V, i.e., an strict sequence of subspaces*

$$V = V_0 \supsetneq V_1 \supsetneq \cdots \supsetneq V_l = \{0\}$$

If we have $d_i = \dim(V_i)$, then

$$n = \dim(V) = d_0 > d_1 > \cdots > d_l = 0$$

The signature of the flag is the sequence $(d_0, \ldots, d_l)$. A flag over V is said to be a full flag if $d_{i+1} = d_i + 1$ for all i.

Given a flag over a space V, we can specify some additional information prescribing a system of weights over the filtration, i.e., assigning for each $i = 1, \ldots, l$, real numbers $0 \leq \alpha_i < 1$ such that

$$0 \leq \alpha_1 < \cdots < \alpha_l < 1$$

A parabolic vector bundle will be a vector bundle over X together with a parabolic structure over the punctures

Definition 4.1.2. *Let X be a smooth projective curve over $\mathbb{C}$ of genus $g \geq 2$. Let D be a finite set of $n \geq 1$ distinct points of X. A parabolic vector bundle over X is a holomorphic vector bundle of rank r together with a weighted flag on the fiber E_x over each $x \in D$ called parabolic structure, i.e.*

$$E_x = E_{x,0} \supsetneq E_{x,1} \supsetneq \cdots \supsetneq E_{x,l(x)} = \{0\}$$
$$0 \leq \alpha_1(x) < \cdots < \alpha_{l(x)}(x) < 1$$

We denote $\alpha = \{(\alpha_1(x), \ldots, \alpha_{l(x)}(x))\}_{x \in D}$ to the system of weights corresponding to a fixed parabolic structure. A parabolic vector bundle is said to be full flag if it has a weighted full flag on every fiber E_x over each $x \in D$, i.e.

$$E_x = E_{x,0} \supsetneq E_{x,1} \supsetneq \cdots \supsetneq E_{x,r} = \{0\}$$

$$0 \leq \alpha_1(x) < \cdots < \alpha_r(x) < 1$$

We will only work with full flag parabolic vector bundles, so we will refer to them simply as parabolic vector bundles.

The system of weights of the parabolic structure of a parabolic vector bundle can be described in an alternative more geometric way. A parabolic vector bundle can be defined to be a vector bundle with a full flag decreasing left continuous filtration over the punctures.

Definition 4.1.3. *For each puncture $x \in D$, a decreasing left continuous filtration is a collection of linear subspaces $E_{\alpha,x} \subseteq E_x$ indexed by real $\alpha \geq 0$ such that*

a) For every $\alpha \geq \beta$, $E_{\alpha,x} \subseteq E_{\beta,x}$

b) For every $\alpha > 0$ there exist $\epsilon > 0$ such that $E_{\alpha-\epsilon,x} = E_{\alpha,x}$

c) If z is a local coordinate of X vanishing to order exactly one at x, then for every $\alpha \geq 0$, $E_{\alpha+1,x} = zE_{\alpha,x}$.

Condition (c) implies that the filtration is completely determined by the subspaces $\{E_{\alpha,x}\}_{0 \leq \alpha < 1}$. The first condition just fixes the filtration to be decreasing. Left continuity condition (b) is the key point to understand the relation with the previously defined parabolic structure. It implies that the function $\alpha \mapsto \mathrm{rk}(E_{\alpha,x})$ is left continuous. As it maps $[0,1] \to \{0, \ldots, \mathrm{rk}(E_x)\}$, the set of points over which the function is not continuous must form an increasing finite sequence $0 \leq \alpha_1(x) < \cdots < \alpha_l(x) < 1$ for some l. Thus, there exist a correspondence between left continuous filtrations and weighted flags.

The filtration has an associated grading $\mathrm{Gr}(\{E_{\alpha,x}\})$. In order to restrict the parabolic structures to full flags it is clearly enough to ask $\dim(\mathrm{Gr}(\{E_{\alpha,x}\})) \leq 1$ for all $0 \leq \alpha < 1$. A decreasing left continuous filtration satisfying this property will be also called full flag.

A parabolic vector bundle defined in terms of left continuous filtrations can be described globally using subsheaves of the vector bundle.

Definition 4.1.4. *A parabolic vector bundle over X with parabolic points D is a vector bundle E on X together with a collection of subsheaves $\{E_\alpha\}_{\alpha\geq 0}$ such that*

a) For every $x \in X$, $\{E_{\alpha,x}\}_{\alpha\geq 0}$ is a full flag decreasing left continuous filtration of E_x

b) The support of $\mathrm{Gr}(\{E_\alpha\})$ lies on D

Condition (b) is equivalent to saying that the filtration is trivial at every $x \in U$, so it is globally defined by the filtrations over D.

In order to define the category of parabolic vector bundles completely, once we have described the concept of parabolic vector bundle, we must discuss morphisms between them. Intuitively, a morphism between parabolic vector bundles will be a vector bundle morphism that preserves the corresponding filtrations. The precise statement is easier to formalize in terms of left continuous filtrations.

Definition 4.1.5. *Let (E, E_α) and (F, F_α) be parabolic vector bundles over X with parabolic points D. A morphism of parabolic vector bundles $\varphi : (E, E_\alpha) \to (F, F_\alpha)$ is a morphism of vector bundles $\varphi : E \to F$ such that for every $\alpha \geq 0$*

$$\varphi(E_\alpha) \subseteq F_\alpha$$

We say that (E, E_α) is a parabolic subbundle of (F, F_α) if the inclusion morphism $i : E \to F$ is a morphism of parabolic vector bundles.

It is obvious that not every vector bundle morphism is a parabolic morphism, but every subbundle can be given an induced parabolic structure so that it becomes a parabolic subbundle.

Definition 4.1.6. *Let $F \subseteq E$ be a subbundle of a parabolic vector bundle (E, E_α). Then F can be given a parabolic structure $\{F_\alpha\}$ taking the left continuous filtration $F_{\alpha,x} = E_{\alpha,x} \cap F_x$ for every $x \in D$. We call this structure the induced parabolic structure on F by (E, E_α).*

Proposition 4.1.7. *The induced left continuous filtration from a full flag parabolic structure is full flag.*

Proof. Let us fix a point $x \in D$ and let $\{\alpha_i\}_{i=1}^r$ be the parabolic weights of the left continuous filtration $E_{\alpha,x}$. Let $E_i = E_{\alpha_i,x}$. It is clear that the rank of $F_{\alpha,x}$ can only change at the points α_i. Then Grassman formula gives us

$$\dim(E_i \cap F_x) = \dim(E_i) + \dim(F_x) - \dim(E_i + F_x)$$

$$\dim(E_{i+1} \cap F_x) = \dim(E_{i+1}) + \dim(F_x) - \dim(E_{i+1} + F_x)$$

We know that $\dim(E_i) = \dim(E_{i+1}) + 1$ and $E_{i+1} + F_x \subseteq E_i + F_x$. Thus

$$\begin{aligned}\dim(E_i \cap F_x) - \dim(E_{i+1} \cap F_x) = \dim(E_i) - \dim(E_{i+1}) + \\ \dim(E_{i+1} + F_x) - \dim(E_i + F_x) \leq \dim(E_i) - \dim(E_{i+1}) = 1 \quad (4.1.1)\end{aligned}$$

□

Therefore, the subbundle F acquires the structure of a parabolic vector bundle. By construction of the induced parabolic structure it is clear that for every $x \in D$, $i : F \to E$ fulfills the condition

$$i(F_{\alpha,x}) = E_{\alpha,x} \cap F_x \subseteq E_{\alpha,x}$$

We will see that many of the basic properties of morphisms of vector bundles described in the first chapter hold for morphisms of parabolic vector bundles.

Proposition 4.1.8. *Let $f : (E, E_\alpha) \to (F, F_\alpha)$ be a morphism of parabolic vector bundles. Then*

a) $\mathrm{Ker}(f)$ *is a parabolic vector bundle with the filtration* $(\mathrm{Ker}(f))_\alpha := \mathrm{Ker}(f|_{E_\alpha})$ *and it is a parabolic subbundle of* (E, E_α)*.*

b) $\mathrm{Im}(f)$ *is a parabolic vector bundle with the filtration* $(\mathrm{Im}(f))_\alpha := f(E_\alpha)$ *and it is a parabolic subbundle of* (F, F_α)*.*

Proof. Proposition 1.4.4 states that $\mathrm{Ker}(f)$ and $\mathrm{Im}(f)$ are vector bundles. By construction, they are subbundles of E and F. First of all we will prove that $(\mathrm{Ker}(f)_\alpha)$ and $(\mathrm{Im}(f)_\alpha)$ are parabolic structures. Let $x \in D$ and let $\alpha_i(x)$ be the parabolic weights of E_α at x. It is clear that the rank of $\mathrm{Ker}(f|_{E_{\alpha,x}})$ and $f(E_{\alpha,x})$ can only change at the weights $\alpha_i(x)$. Let $E_i := E_{\alpha_i(x),x}$ and $f_i = f|_{E_i} : E_i \to F_x$.

As $E_{i+1} \subseteq E_i$, then $f(E_{i+1}) \subseteq f(E_i)$. E_α is left continuous. Therefore $(\mathrm{Im}(f))_\alpha$ is a decreasing left continuous filtration. Similarly, if $x \in \mathrm{Ker}(f_{i+1})$ then $x \in E_{i+1}$ and $f(x) = 0$. Thus, $x \in E_i$ and $f(x) = 0$, so $x \in \mathrm{Ker}(f_i)$. Therefore, $\mathrm{Ker}(f_{i+1}) \subseteq \mathrm{Ker}(f_i)$, and we get that $(\mathrm{Ker}(f))_\alpha$ is a decreasing left continuous filtration. We have to prove that both are full flag filtrations. As E_α is full flag, for every i we get that $\dim(E_i) = \dim(E_{i+1}) + 1$. Therefore

$$\dim(\mathrm{Ker}(f_{i+1})) + \dim(\mathrm{Im}(f_{i+1})) = \dim(E_{i+1})$$

$$\dim(\mathrm{Ker}(f_i)) + \dim(\mathrm{Im}(f_i)) = \dim(E_i) = \dim(E_{i+1}) + 1$$

As $\dim(\mathrm{Ker}(f_{i+1})) \leq \dim(\mathrm{Ker}(f_i))$ and $\dim(\mathrm{Im}(f_{i+1})) \leq \dim(\mathrm{Im}(f_i))$ we obtain that

$$\dim(\mathrm{Ker}(f_i)) \leq \dim(\mathrm{Ker}(f_{i+1})) + 1$$

$$\dim(\mathrm{Im}(f_i)) \leq \dim(\mathrm{Im}(f_{i+1})) + 1$$

On the other hand, as f is a morphism of parabolic vector bundles, for every $\alpha \geq 0$, $f(E_\alpha) \subseteq F_\alpha$, so $\mathrm{Im}(f)$ is a parabolic subbundle of F. Moreover, $\mathrm{Ker}(f|_{E_\alpha}) \subseteq E_\alpha$, so $\mathrm{Ker}(f)$ is a parabolic subbundle of E. □

With this definition of kernel and image of a parabolic vector bundle morphism, taking the zero object of the category as the trivial vector bundle together with the trivial zero filtration, there exist the concept of an exact sequence of parabolic vector bundles. In order to understand the structure of these sequences better, we introduce the following proposition.

Proposition 4.1.9. *The sequence of parabolic vector bundles over X, parabolic over D*

$$0 \longrightarrow (A, A_\alpha) \xrightarrow{f} (B, B_\alpha) \xrightarrow{g} (C, C_\alpha) \longrightarrow 0 \tag{4.1.2}$$

is exact if and only if it is exact as a sequence of vector bundles and for every $\alpha \geq 0$ and $x \in D$, the sequence

$$0 \longrightarrow A_\alpha \xrightarrow{f|_{A_\alpha}} B_\alpha \xrightarrow{g|_{B_\alpha}} C_\alpha \longrightarrow 0 \tag{4.1.3}$$

is exact.

Proof. First of all, as all the morphisms in the exact sequence (4.1.2) are morphisms of parabolic vector bundles, the image of each filtration under the correspondent morphism lies inside the filtration of the image object. Thus, the sequence (4.1.3) is well defined.

The sequence (4.1.2) is exact if and only if the following equalities of parabolic vector bundles hold

$$\begin{cases} \mathrm{Im}(0 \to (A, A_\alpha)) = \mathrm{Ker}(f) \\ \mathrm{Im}(f) = \mathrm{Ker}(g) \\ \mathrm{Im}(g) = \mathrm{Ker}((C, C_\alpha) \to 0) \end{cases} \tag{4.1.4}$$

Two vector bundles are equal if and only if their underlying vector bundles are equal and the corresponding filtrations agree for every $x \in D$ and every $\alpha \geq 0$. Considering the vector bundle equalities resulting from dropping the parabolic structure in the previous equalities we precisely obtain the conditions needed for (4.1.2) to be exact as a sequence of vector bundles.

On the other hand, by definition of image and kernel given by 4.1.8, for each $x \in D$ and each$\alpha \geq 0$, the equalities of filtrations given by (4.1.4) result in

$$\begin{cases} 0 = \mathrm{Im}(0 \to A_\alpha) = \mathrm{Ker}(f|_{A_\alpha}) \\ \mathrm{Im}(f|_{A_\alpha}) = \mathrm{Ker}(g|_{B_\alpha}) \\ \mathrm{Im}(g|_{B_\alpha}) = \mathrm{Ker}(C_\alpha \to 0) = 0 \end{cases} \tag{4.1.5}$$

Thus, the filtrations satisfy equation (4.1.4) if and only if the sequence (4.1.3) is exact. □

As a corollary we get the usual exact sequence for the image and kernel of a morphism.

Corollary 4.1.10. *Let $f : (E, E_\alpha) \to (F, F_\alpha)$ be a morphism of parabolic vector bundles. Then the sequence*

$$0 \longrightarrow (\mathrm{Ker}(f), (\mathrm{Ker}(f))_\alpha) \xrightarrow{i} (E, E_\alpha) \xrightarrow{f} (\mathrm{Im}(f), (\mathrm{Im}(f))_\alpha) \longrightarrow 0$$

is exact.

Proof. The sequence is clearly exact as a vector bundle sequence, so it is enough to prove that the corresponding induced filtrations form an exact sequence for every $x \in$

D and every $\alpha \geq 0$. This is followed directly by the definition of the corresponding filtrations for the image and kernel of f, because

$$0 \to (\mathrm{Ker}(f))_\alpha = \mathrm{Ker}(f|_{E_{\alpha,x}}) \to E_{\alpha,x} \to \mathrm{Im}(f|_{E_{\alpha,x}}) = (\mathrm{Im}(f))_\alpha \to 0$$

is exact. □

Using the previous propositions on exact sequences we can define the quotient of a parabolic vector bundle (E, E_α) by a subbundle (E', E'_α) as the unique parabolic vector bundle $(E/E', E_\alpha/E'_\alpha)$ such that the sequence

$$0 \to (E', E'_\alpha) \to (E, E_\alpha) \to (E/E', E_\alpha/E'_\alpha) \to 0$$

is exact as a sequence of parabolic vector bundles.

Finally, we will give another interpretation of parabolic bundles in terms of sheaf extensions. Let $i : U \hookrightarrow X$ be the inclusion. Let E be a vector bundle defined on U. We know that E is a locally free sheaf of $\mathcal{O}_U$-modules. Let us now consider the sheaf $i_*\mathcal{O}_U$ on X. There exist a canonical isomorphism

$$i_*\mathcal{O}_U \cong \mathcal{O}_X\left(\sum_{x\in D} \infty \cdot x\right)$$

where $\mathcal{O}_X\left(\sum_{x\in D} \infty \cdot x\right)$ is the sheaf of local meromorphic functions on X with poles over D of arbitrary finite order. Now, we can consider the sheaf i_*E on X. It is naturally a sheaf of $i_*\mathcal{O}_U$-modules. As E was finitely generated, i_*E will be finitely generated as a $i_*\mathcal{O}_U$-module. On the other hand, as $\mathcal{O}_X$ is a subsheaf of $i_*\mathcal{O}_U$, i_*E is a sheaf of $\mathcal{O}_X$-modules. Nevertheless, it is clear that, in general, i_*E is not coherent as a sheaf of $\mathcal{O}_X$-modules. A parabolic subbundle can be seen as a filtration of i_*E by a left continuous increasing filtration $\{E_\alpha\}_{\alpha\geq 0}$ such that for each α, E_α is a coherent subsheaf of $\mathcal{O}_X$-modules of i_*E.

As we will see later on, the parabolic weights allow us to control in some sense the "divergence rate" of the structures over the parabolic points in the directions marked by the filtration. We will consider parabolic vector bundles on which we have prescribed this rates, i.e., on which we have prescribed the system of weights of the parabolic structure over each puncture.

4.2 Parabolic stabilty

Once we have fixed the system of weights for a parabolic vector bundle, we will define a version of the degree and the slope of the vector bundle that takes into account its parabolic structure.

Definition 4.2.1. *Let α be a fixed parabolic structure and let E be a parabolic vector bundle over X . The parabolic degree of E is defined as*

$$\mathrm{pardeg}(E) = \deg(E) + \sum_{x\in D} \sum_{i=1}^{\mathrm{rk}(E)} \alpha_i(x)$$

We define the parabolic slope of a parabolic vector bundle as

$$\operatorname{par}\mu(E) = \frac{\operatorname{pardeg}(E)}{\operatorname{rk}(E)}$$

If a parabolic vector bundle is defined in terms of a left continuous filtration, we can give the following equivalent definition of the parabolic degree

$$\operatorname{pardeg}(E) = \deg(E) + \sum_{x\in D}\sum_{\alpha\geq 0} \alpha \dim(\operatorname{Gr}_\alpha(E_{\alpha,x}))$$

As $\dim(\operatorname{Gr}_\alpha(E_{\alpha,x})) \leq 1$ for all $\alpha \geq 0$ and it is one exactly at the parabolic weights $\alpha_i(x)$, the definitions are clearly equivalent. The first important property of the parabolic degree is the additivity.

Lemma 4.2.2. *For every exact sequence of parabolic vector bundles*

$$0 \longrightarrow (A, A_\alpha) \xrightarrow{f} (B, B_\alpha) \xrightarrow{g} (C, C_\alpha) \longrightarrow 0$$

we have that $\operatorname{pardeg}(A) + \operatorname{pardeg}(C) = \operatorname{pardeg}(B)$.

Proof. Proposition 4.1.9 proves that for every $x \in D$ and every $\alpha \geq 0$, there is an exact sequence

$$0 \to A_{\alpha,x} \to B_{\alpha,x} \to C_{\alpha,x} \to 0$$

Thus, for every $\alpha \geq 0$, $\dim(A_{\alpha,x}) + \dim(C_{\alpha,x}) = \dim(B_{\alpha,x})$. Let $x \in D$ be a fixed point. Let a_i, b_i and c_i be the parabolic weights of $A_{\alpha,x}$, $B_{\alpha,x}$ and $C_{\alpha,x}$ respectively. Let $\{t_i\} = \{a_i\} \cup \{b_i\} \cup \{c_i\}$. Then for every j such that $a_i < t_j < t_{j+1} \leq a_{i+1}$ we have that

$$\dim(A_{t_j,x}) = \dim(A_{t_{j+1},x})$$

Thus, $t_j \left(\dim(A_{t_j,x}) - \dim(A_{t_{j+1},x})\right) = 0$, and we get that

$$\sum_{a_i\leq t_j<a_{i+1}} t_j \left(\dim(A_{t_j,x}) - \dim(A_{t_{j+1},x})\right) = a_i$$

Thus,

$$\sum_j t_j \left(\dim(A_{t_j,x}) - \dim(A_{t_{j+1},x})\right) = \sum_{i=1}^{\operatorname{rk}(A)} a_i$$

We have similar equations for A and B. For each j we have that

$$\dim(A_{t_j,x}) - \dim(A_{t_{j+1},x}) + \dim(C_{t_j,x}) - \dim(C_{t_{j+1},x}) = \dim(B_{t_j,x}) - \dim(B_{t_{j+1},x})$$

Multiplying both sides by t_j and adding on j we get that

$$\sum_{i=1}^{\operatorname{rk}(A)} a_i + \sum_{i=1}^{\operatorname{rk}(C)} c_i = \sum_j t_j \left(\dim(A_{t_j,x}) - \dim(A_{t_{j+1},x}) + \dim(C_{t_j,x}) - \dim(C_{t_{j+1},x})\right)$$

$$= \sum_j t_j \left(\dim(B_{t_j,x}) - \dim(B_{t_{j+1},x})\right) = \sum_{i=1}^{\operatorname{rk}(B)} b_i \quad (4.2.1)$$

As vector bundle degree is additive, we get that $\deg(A) + \deg(C) = \deg(B)$. Adding both equations for every $x \in D$we obtain that

$$\operatorname{pardeg}(A) + \operatorname{pardeg}(C) = \deg(A) + \sum_{x\in D}\sum_{i=1}^{\operatorname{rk}(A)} a_i(x) + \deg(C) + \\ \sum_{x\in D}\sum_{i=1}^{\operatorname{rk}(C)} c_i(x) = \deg(B) + \sum_{x\in D}\sum_{i=1}^{\operatorname{rk}(B)} b_i(x) = \operatorname{pardeg}(B) \quad (4.2.2)$$

□

Once we have proved that the parabolic degree is additive, the proof of 2.4.2 hold for parabolic vector bundles, leading to the following proposition

Proposition 4.2.3. *Let* $0 \longrightarrow (A, A_\alpha) \xrightarrow{f} (B, B_\alpha) \xrightarrow{g} (C, C_\alpha) \longrightarrow 0$ *be a exact sequence of parabolic vector bundles. Then either* $\operatorname{par}\mu(A) \leq \operatorname{par}\mu(B) \leq \operatorname{par}\mu(C)$ *or* $\operatorname{par}\mu(A) \geq \operatorname{par}\mu(B) \geq \operatorname{par}\mu(C)$. *Moreover, if some of the inequalities are strict all the inequalities are strict.*

We would like to define a moduli space for parabolic vector bundles in a similar way to the moduli space of semistable vector bundles of fixed rank and determinant. Thus, it is necessary to define a notion of stability.

Definition 4.2.4. *A parabolic bundle is said to be parabolicaly (semi)stable if for all parabolic subbundles* $F \subsetneq E$ *with the induced parabolic structure we have*

$$\operatorname{par}\mu(F) < \operatorname{par}\mu(E) \qquad (\leq) \tag{4.2.3}$$

Notice that we are only testing subbundles with the induced parabolic structure in order to define stability. This is, in fact, equivalent to test stability for every subbundle. Let (F, F_α) be a parabolic subbundle of (E, E_α). Then we have that $F_\alpha \subseteq E_\alpha$ for every $\alpha \geq 0$. Thus, $F_\alpha \subseteq E_\alpha \cap F$ and (F, F_α) is a parabolic subbundle of $(F, E_\alpha \cap F)$. Let us prove that the parabolic slope of (F, F_α) is less than the parabolic slope of $(F, E_\alpha \cap F)$. As their underlying vector bundles are the same, their rank and degree are equal, so it is enough to prove that the sum of the parabolic weights of F_α is less than the sum of the parabolic weights of F_α.

The i-th parabolic weight of $F_{\alpha,x}$ is $\alpha_i = \max(\alpha : \dim(F_{\alpha,x}) = \operatorname{rk}(F) - i + 1)$. Similarly, the i-th parabolic weight of $E_{\alpha,x} \cap F_x$ is $\alpha_i' = \max(\alpha : \dim(E_{\alpha,x} \cap F_x) = \operatorname{rk}(F) - i + 1)$. As $\dim(F_{\alpha,x}) \leq \dim(E_{\alpha,x} \cap F_x)$ for every α, we get that $\alpha_i \leq \alpha_i'$ for every $i = 1, \ldots, \operatorname{rk}(F)$. Therefore $\sum_{i=1}^{\operatorname{rk}(F)} \alpha_i \leq \sum_{i=1}^{\operatorname{rk}(F)} \alpha_i'$.

On the other hand, we are only testing the parabolic condition for subbundles and not for subsheaves, just as for regular vector bundles. The reason, again, is that we can avoid considering subbundles thanks to their saturations. In the case of regular vector bundles the saturation of a subsheaf is a subbundle with higher slope. If we now take a subsheaf E' of a parabolic vector bundle (E, E_α), forgetting about the parabolic structure we can consider its saturation $\overline{E'}$. In chapter 1 we proved that it is a subbundle of E of the same rank and higher degree such that has E' as a subsheaf.

Let us now study the corresponding parabolic structure for the subsheaf and its saturation. We can define a parabolic subsheaf in a similar way to parabolic subbundle

Definition 4.2.5. *Let (E, E_α) be a parabolic vector bundle. A parabolic subsheaf (F, F_α) is a subsheaf that is a parabolic vector bundle, such that for every $x \in D$, the inclusion morphism $i : F \to E$ fulfills*

$$i(F_{\alpha,x}) \subseteq E_{\alpha,x}$$

The difference between a subsheaf and a subbundle lies at the variation of the rank of the immersion i. As the parabolic structures are defined on a finite set of points, one can impose the condition of i preserving the filtrations at the parabolic points without considering the rank of the morphism.

Let now (E', E'_α) be a subsheaf of (E, E_α) and let $\overline{E'}$ be the saturation of E'. As (E', E'_α) is a parabolic subsheaf of (E, E_α), for every $x \in D$ we get that

$$i(E'_{\alpha,x}) \subseteq E_{\alpha,x}$$

On the other side, as E' is a subsheaf of $\overline{E'}$, we get that $i(E'_{\alpha,x}) \subseteq \overline{E'}$. Thus

$$i(E'_{\alpha,x}) \subseteq E_{\alpha,x} \cap \overline{E'}$$

As $\overline{E'}$ is a subbundle of E, it can be given the induced parabolic structure $E_{\alpha,x} \cap \overline{E'}$ to make it a parabolic subbundle. The previous equation tells us that with this structure, (E', E'_α) is a parabolic subsheaf of $(\overline{E'}, E_\alpha \cap \overline{E'})$, and we already know that the parabolic degree of $\overline{E'}$ among all possible structures that make it a parabolic subbundle of (E, E_α) is maximum when we take the induced parabolic structure. Thus, we get that the saturation of E' with the induced parabolic structure has greater slope than E'. Therefore, it is only necessary to check the stability condition in the terms exposed previously.

The parabolic stability of a parabolic vector bundle has similar properties as the usual stability.

Proposition 4.2.6. *Let (E, E_α) and (F, F_α) be two parabolically semistable vector bundles. If* $\operatorname{par}\mu(E) > \operatorname{par}\mu(F)$ *then* $\operatorname{Hom}_{par}((E, E_\alpha), (F, F_\alpha)) = 0$.

Proposition 4.2.7. *Let (E, E_α) and (F, F_α) be two stable vector bundles. If* $\operatorname{par}\mu(E) = \operatorname{par}\mu(F)$ *then every nonzero parabolic morphism is an isomorphism.*

The proofs are completely analogous to those of the corresponding statements for vector bundles. Moreover, one can define Jordan-Hölder and Harder-Narasimhan filtrations for parabolic vector bundles exactly the same way as one builds the corresponding filtrations for vector bundles.

Proposition 4.2.8 (Jordan-Hölder filtration for parabolic vector bundles)**.** *Let (E, E_α) be a parabolically semistable vector bundle. There exist a filtration of parabolic subbundles*

$$0 = (E_0, E_{0,\alpha}) \subset (E_1, E_{1,\alpha}) \subset \cdots \subset (E_n, E_{n,\alpha}) = (E, E_\alpha)$$

with $(E_i/E_{i-1}, E_{i,\alpha}/E_{i-1,\alpha})$ parabolically stable and $\operatorname{par}\mu(E_i/E_{i-1}) = \operatorname{par}\mu(E)$. *Moreover, the parabolic grading* $\operatorname{Gr}(E, E_\alpha)$ *is unique up to isomorphism.*

Proposition 4.2.9 (Harder-Narashimhan filtration for parabolic vector bundles)**.** *Let (E, E_α) be a parabolic vector bundle. There exist a filtration of parabolic subbundles*

$$0 = (E_0, E_{0,\alpha}) \subset (E_1, E_{1,\alpha}) \subset \cdots \subset (E_n, E_{n,\alpha}) = (E, E_\alpha)$$

with $(E_i/E_{i-1}, E_{i,\alpha}/E_{i-1,\alpha})$ parabolically semistable and $\operatorname{par}\mu(E_i/E_{i-1}) > \operatorname{par}\mu(E_{i+1}/E_i)$. Moreover, the filtration is unique up to isomorphism.

4.3 Moduli of semistable parabolic vector bundles

Using an analogous moduli problem to the one used for vector bundles, Mehta and Seshadri [MS80] proved that there exist a coarse moduli space $\mathcal{M}(r, d, \alpha, X)$ representing the classes of semistable parabolic vector bundles of rank r and degree d with fixed system of weights α over X. Notice that since the system of weights is fixed, prescribing the parabolic degree of a parabolic vector bundle is equivalent to fixing the degree of its underlying vector bundle. They also proved the following proposition

Proposition 4.3.1. *Let X be a smooth projective curve of genus $g \geq 2$. The scheme $\mathcal{M}(r, d, \alpha, X)$ is a normal projective variety of dimension*

$$r^2(g-1) + 1 + \frac{n(r^2 - r)}{2} \quad .$$

Intuitively, the first summands come from the dimension of the moduli $\mathcal{M}(r, d, X)$, whereas the last summand comes from the fact that we are choosing n full flags over each parabolically semistable vector bundle. Similarly to the moduli space of semistable vector bundles, let $\mathcal{M}^s(r, d, \alpha, X)$ be the subvariety of $\mathcal{M}(r, d, \alpha, X)$ corresponding to parabolically stable vector bundles. Then $\mathcal{M}^s(r, d, \alpha, X)$ is smooth.

Let us consider a line bundle ξ over X of degree d. Similarly to the moduli of vector bundles over X, we can consider the subvariety of $\mathcal{M}(r, d, \alpha, X)$ corresponding to semistable vector bundles with determinant ξ.

Let $\mathcal{M}(r, \alpha, \xi, X)$ be the moduli space of semi-stable parabolic vector bundles on X of rank r with weight system α together with an isomorphism $\bigwedge^r E \cong \xi$. We will omit the curve X whenever it is clear. It is a projective scheme of dimension

$$\dim(\mathcal{M}(r, \alpha, \xi)) = (g-1)(r^2 - 1) + \frac{n(r^2 - r)}{2}$$

Moreover, Boden and Yokogawa proved in [BY99] that $\mathcal{M}(r, \alpha, \xi)$ is rational whenever the genus of the curve is greater than one. Again, we will denote by $\mathcal{M}^s(r, \alpha, \xi)$ the open subvariety parametrizing the parabolicaly stable bundles. As $\mathcal{M}^s(r, d, \alpha)$ is smooth, this open subvariety lies inside the smooth locus of $\mathcal{M}(r, \alpha, \xi)$.

Chapter 5

Parabolic Higgs bundles

In the previous chapter we have described parabolic vector bundles as a tool that allows us to define vector bundles over noncompact curves precisely, at the same time that we prescribe certain parabolic structure at the punctures of the curve.

We will use this notion to extend the definition of Higgs bundles given in chapter 3 to punctured curves. Our goal is to describe Higgs bundles that are smooth over a punctured curved but which may have poles of order one at the punctures. Let X be a smooth complex projective curve of genus $g \geq 2$. Let D be a set of punctures over D and let α be a system of weights of rank $r \geq 2$ over D. Let K be the canonical bundle over X.

5.1 Parabolic Higgs field

The key to describe a Higgs bundle is the Higgs field. In the nonparabolic situation, the Higgs field is a vector bundle morphism $\Phi : E \to E \otimes K$. On the other hand, parabolic Higgs bundles will be allowed to have poles of order one at the parabolic points, i.e., the Higgs field will be a meromorphic vector bundle morphism $\Phi : E \to E \otimes K$ with poles of order at most one on D.

If K is the canonical bundle, let $K(D)$ be the line bundle $K \otimes \mathcal{O}_X(D)$. This line bundle parametrizes meromorphic 1-forms with poles of order at most one at the points in D. Thus, meromorphic vector bundle morphisms $\Phi : E \to E \otimes K$ with poles of order at most one over D are equivalent to holomorphic vector bundle morphisms $\Phi : E \to E \otimes K(D)$.

The next step is to control the behavior of the singularities of the Higgs field at the parabolic points. We will acknowledge this through the use of a parabolic structure over the punctures. For every $x \in D$, the Poincaré adjunction formula states that $\mathcal{O}_X(D)|_x \cong T_xX$, so

$$K(D)|_x = K|_x \otimes \mathcal{O}_X(D)_x \cong T_x^*X \otimes T_xX \cong \mathbb{C}$$

Then for every $x \in D$, Φ induces an endomorphism of every fiber E_x

$$\Phi|_{E_x} : E_x \to E_x \otimes K(D)|_x \cong E_x$$

If (E, E_α) is a parabolic vector bundle, then there exist an induced parabolic structure on $E \otimes K(D)$ taking the filtration $E_{\alpha,x} \otimes K(D)|_x$ for every $x \in D$. Therefore it is natural to ask parabolic Higgs fields to be parabolic homomorphisms $E \to E \otimes K(D)$ with respect to the parabolic filtration of E and the induced parabolic filtration on $E \otimes K(D)$.

Taking all this into account, we define a parabolic Higgs bundle as follows.

Definition 5.1.1. *A parabolic Higgs bundle (E, Φ) is a parabolic vector bundle E together with a homomorphism called Higgs field*

$$\Phi : E \longrightarrow E \otimes K(D)$$

which is a parabolic homomorphism, i.e. for each $x \in D$ the homomorphism induced in the filtration over the fiber E_x satisfies

$$\Phi(E_{x,i}) \subseteq E_{x,i} \otimes K(D)|_x \quad .$$

As a simplification, we will only consider parabolic Higgs fields with zero spectrum at the parabolic points. We will see that this is equivalent to the Higgs field being strictly parabolic in the following sense,

Definition 5.1.2. *Let $(E, \{E_{i,x}\})$ be a parabolic vector bundle. A strongly parabolic endomorphism of E is an endomorphism $\Phi : E \to E$ such that for every point $x \in D$,*

$$\Phi(E_{x,i}) \subseteq E_{x,i+1} \quad .$$

Analogously, we say that an endomorphism is non-strongly parabolic if it satisfies

$$\Phi(E_{x,i}) \subseteq E_{x,i} \quad .$$

Denote by $\mathrm{SParEnd}(E)$ the sheaf of strongly parabolic endomorphisms and by $\mathrm{ParEnd}(E)$ the sheaf of non-strongly parabolic endomorphisms.

We say that a parabolic Higgs bundle is strongly parabolic if the Higgs field is a strongly parabolic homomorphism $\varphi : E \to E \otimes K(D)$, i.e., for each $x \in D$ the homomorphism induced in the filtration over E_x satisfies

$$\Phi(E_{x,i}) \subseteq E_{x,i+1} \otimes K(D)|_x \quad .$$

Proposition 5.1.3. *Let $(E, \{E_{i,x}\})$ be a parabolic vector bundle. An endomorphism $\Phi : E \to E$ is strictly parabolic if and only if for every $x \in D$, the linear morphism $\Phi : E_x \to E_x$ obtained restricting Φ to E_x has zero spectrum.*

Proof. Let us fix a point $x \in D$ and let $r > 0$ be the rank of E. Let $\Phi : E_x \to E_x$ be a parabolic morphism at x. Let us choose a basis of E_x compatible with the filtration $\{E_{i,x}\}$, i.e., a set of r linearly independent elements of E_x, say $\{v_i\}$, such that $E_{i,x} = \mathrm{span}(v_1, \ldots, v_{r-i})$. As Φ is parabolic, for every i, $v_k \in E_{r-k}$, so $\Phi(v_k) \in E_{r-k,x} = \mathrm{span}(v_1, \ldots, v_k)$. Thus, in the basis $\{v_i\}$, Φ is given by a lower triangular matrix, whose spectrum is given by the entries at the diagonal. Thus, the spectrum of Φ is trivial if and only if the entries at the diagonal are trivial, i.e., if and only if $\Phi(v_k) \in \mathrm{span}(v_1, \ldots, v_{k-1})$ for every $k = 1, \ldots, r$.

Therefore, if Φ has trivial spectrum, for every k, $\Phi(v_k) \in \operatorname{span}(v_1, \dots, v_{k-1})$, so we get that for every k,

$$\begin{aligned}\Phi(E_{r-k}) = \Phi(\operatorname{span}(v_1, \dots, v_k)) = \operatorname{span}(\Phi(v_1), \dots, \Phi(v_k))\\ \subseteq \operatorname{span}(v_1, \dots, v_{k-1}) = E_{r-k+1}\end{aligned}$$

and we get that Φ is strictly parabolic.

Let us suppose that Φ has a nontrivial eigenvalue $\lambda \neq 0$. Then there exit a nonzero eigenvector $v \in E_x$ such that $\Phi(v) = \lambda v$. As $\{E_{i,x}\}$ is a filtration, there exist an index j such that $v \in E_{j,x} \backslash E_{j+1,x}$. Nevertheless, as Φ is strictly parabolic $\Phi(v) = \lambda v \in E_{j+1,x}$. But it is impossible that $\lambda v \in E_{j+1,x}$ and $v \notin E_{j+1,x}$, so Φ has a trivial spectrum. □

Later on, we are going to analyze Higgs bundles over parabolic vector bundles with a prescribed determinant in order to obtain SL-bundles. In order to do so, we will only consider Higgs bundles with traceless Higgs fields. The situation is completely analogous to the one described back in chapter 3.

5.2 Parabolic Higgs bundles stability

In this section we will combine the notion of stability for Higgs bundles given in chapter 3 and the comments referring parabolic stability made in chapter 4 in order to obtain a suitable notion of stability for the parabolic Higgs bundles scenario.

Definition 5.2.1. *A parabolic subbundle $F \subset E$ is said to be Φ-invariant if $\Phi(F) \subseteq F \otimes K(D)$. A parabolic Higgs bundle is called (semi)stable if the stability slope condition*

$$\operatorname{par}\mu(F) < \operatorname{par}\mu(E) \quad (\leq)$$

holds for every Φ-invariant parabolic subbundle $F \subsetneq E$, $F \neq \{0\}$ with the induced parabolic structure of E.

As we discussed in the previous chapter, it is only necessary to check the parabolic stability condition for parabolic subbundles with the induced parabolic structure, as every other subbundle presents a lower parabolic slope. Comments made in chapter 2 on the stability condition for subsheaves instead of subbundles also hold in the case of parabolic Φ-invariant subbundles, as we have the following lemma.

Lemma 5.2.2. *Let (E', E_α) be a parabolic Φ-invariant subsheaf of (E, E_α). Then the saturation of (E', E'_α), $(\overline{E'}, E_\alpha \cap \overline{E'})$ is a parabolic Φ-invariant subbundle of (E, E_α) of higher parabolic slope such that (E', E'_α) is a parabolic Φ-invariant subsheaf of $(\overline{E'}, E_\alpha \cap \overline{E'})$.*

The proof of the lemma is completely analogous to that of lemma 3.1.3, taking into account that the remark about the parabolic structure of the saturation of a parabolic subsheaf given after the definition 4.2.5 guaranties that the saturation of a parabolic subsheaf is a parabolic subbundle.

As the Higgs field of a parabolic Higgs bundle is required to be a morphism of parabolic Higgs bundles, then all the regularity conditions derived from the existence of the parabolic structure on the punctures hold immediately as a consequence of the propositions given in chapter 4. Thus, the definition of stability given is completely analogous to the one used for regular Higgs bundles, and one can expect that some of the classical theorems for Higgs bundles still hold.

Proposition 5.2.3 (Jordan-Hölder filtration for parabolic Higgs bundles)**.** *Let (E, E_α, Φ) be a semistable parabolic Higgs bundle. There exist a filtration of parabolic Φ-invariant subbundles*

$$0 = (E_0, E_{0,\alpha}) \subset (E_1, E_{1,\alpha}) \subset \cdots \subset (E_n, E_{n,\alpha}) = (E, E_\alpha)$$

with $(E_i/E_{i-1}, E_{i,\alpha}/E_{i-1,\alpha}, \Phi)$ parabolically stable and $\operatorname{par}\mu(E_i/E_{i-1}) = \operatorname{par}\mu(E)$. Moreover, the parabolic grading $\operatorname{Gr}(E, E_\alpha)$ is unique up to isomorphism.

Proposition 5.2.4 (Harder-Narashimhan filtration for parabolic Higgs bundles)**.** *Let (E, E_α, Φ) a parabolic Higgs bundle. There exist a filtration of parabolic Φ-invariant subbundles*

$$0 = (E_0, E_{0,\alpha}) \subset (E_1, E_{1,\alpha}) \subset \cdots \subset (E_n, E_{n,\alpha}) = (E, E_\alpha)$$

with $(E_i/E_{i-1}, E_{i,\alpha}/E_{i-1,\alpha}, \Phi)$ parabolically semistable and $\operatorname{par}\mu(E_i/E_{i-1}) > \operatorname{par}\mu(E_{i+1}/E_i)$. Moreover, the filtration is unique up to isomorphism.

Therefore, there exist a notion of S-equivalence for parabolic Higgs bundles that will be the natural equivalence relation needed in order to build the moduli space of semistable parabolic Higgs bundles.

5.3 Moduli space of parabolic Higgs bundles

We denote by $\mathcal{M}_{Higgs}(r, \alpha, \xi)$ the moduli space of parabolicaly semi-stable Higgs bundles of rank r and weight system α such that the Higgs field Φ is strictly parabolic and traceless, together with an isomorphism $\bigwedge^r E \cong \xi$. In [BY96], Boden and Yokogawa constructed this coarse moduli space from an analogous moduli problem as the one described for Higgs bundles. The existence of a coarse moduli space corresponding to stable parabolic Higgs bundles was previously proved by Konno in [Kon93] using analytic techniques. It is an irreducible normal projective variety of dimension

$$\dim(\mathcal{M}_{Higgs}(r, \alpha, \xi)) = 2(g-1)(r^2-1) + n(r^2-r)$$

We will also call $\mathcal{M}^{sm}_{Higgs}(r, \alpha, \xi)$ the smooth locus of $\mathcal{M}_{Higgs}(r, \alpha, \xi)$. Finally, let $\mathcal{M}^{ns}_{Higgs}(r, \alpha)$ be the moduli space of non-strictly parabolic semistable Higgs bundles of rank r and weight system α.

Similarly to the regular Higgs bundles scenario, there is a natural embedding

$$i : \mathcal{M}(r, \alpha, \xi) \hookrightarrow \mathcal{M}_{Higgs}(r, \alpha, \xi) \tag{5.3.1}$$

defined by $E \mapsto (E,0)$. Let $\mathcal{M}^{st}_{Higgs}(r,\alpha,\xi)$ be the locus of Higgs bundles (E,Φ) whose underlying vector bundle E is parabolicaly stable. It is an open dense subset of $\mathcal{M}_{Higgs}(r,\alpha,\xi)$. Let

$$\mathrm{pr}_E : \mathcal{M}^{st}_{Higgs}(r,\alpha,\xi) \longrightarrow \mathcal{M}^s(r,\alpha,\xi) \tag{5.3.2}$$

be the forgetful map defined by $(E,\Phi) \to E$. By deformation theory, the tangent space at $[E]$, $T_{[E]}\mathcal{M}^s(r,\alpha,\xi)$ is isomorphic to $H^1(X, \mathrm{ParEnd}(E))$. By the parabolic version of Serre duality,

$$H^1(X, \mathrm{ParEnd}(E))^* \cong H^0(X, \mathrm{SParEnd}(E) \otimes K(D))$$

and hence, the Higgs field is an element of the cotangent bundle $T^*_{[E]}\mathcal{M}^s(r,\alpha,\xi)$ and one has a canonical isomorphism

$$\mathcal{M}^{st}_{Higgs}(r,\alpha,\xi) \xrightarrow{\sim} T^*\mathcal{M}^s(r,\alpha,\xi) \tag{5.3.3}$$

of varieties over $\mathcal{M}^s(r,\alpha,\xi)$.

5.4 Hitchin space

We will now introduce the definition of the Hitchin map and the Hitchin space for non-strictly parabolic Higgs bundles. Let $S = \mathbb{V}(K(D))$ be the total space of the line bundle K(D), let

$$p : S = \underline{\mathrm{Spec}}\, \mathrm{Sym}^\bullet(K^{-1} \otimes \xi(D)^{-1}) \longrightarrow X$$

be the projection, and $x \in H^0(S, p^*(K(D)))$ be the tautological section. The characteristic polynomial of a Higgs field

$$\det(x \cdot \mathrm{id} - p^*\Phi) = x^r + \tilde{s_1}x^{r-1} + \tilde{s_2}x^{r-2} + \cdots + \tilde{s_r}$$

defines sections $s_i \in H^0(X, K^iD^i)$, such that $\tilde{s_i} = p^*s_i$ and K^iD^j denotes the tensor product of the i-th power of K with the j-th power of the line bundle associated to D. We define the Hitchin space as

$$\mathcal{H} = \bigoplus_{i=1}^{r} H^0(K^iD^i) \tag{5.4.1}$$

The Hitchin map is defined as

$$H^{ns} : \mathcal{M}^{ns}_{Higgs}(r,\alpha) \longrightarrow \mathcal{H} \tag{5.4.2}$$

sending each Higgs bundle (E,Φ) to the characteristic polynomial of Φ.

We can now restrict the Hitchin map to $\mathcal{M}_{Higgs}(r,\alpha,\xi)$. We are assuming that Φ is strongly parabolic, therefore the residue at each point of D is nilpotent. This implies that the eigenvalues of Φ vanish at D, so for each $i > 0$ the section s_i belongs to the subspace $H^0(X, K^iD^{i-1}) \subseteq H^0(X, K^iD^i)$. Moreover, in order to fix the determinant, we are asking Φ to be traceless, so $s_1 = 0$ and the image in the Hitchin space lies in

$$\mathcal{H}_0 = \bigoplus_{i=2}^{r} H^0(K^i D^{i-1}) \tag{5.4.3}$$

Therefore, one obtains a map

$$H : \mathcal{M}_{Higgs}(r, \alpha, \xi) \longrightarrow \mathcal{H}_0 \tag{5.4.4}$$

The spectral curve for a point $s \in \mathcal{O}_0$ can be defined essentially the same way as in the case of regular Higgs bundles, and we will also call $H^{-1}(0)$ the nilpotent cone. Some of the propositions stated in chapter 3 still hold.

Proposition 5.4.1. *For $r \geq 2$ and $g \geq 2$, there is a dense open set in $\mathcal{H}$ whose spectral curve is smooth. The same holds for $\mathcal{H}_0$.*

The proof can be found in [GL11, Lemma 3.1]. Similarly, [GL11, Lemma 3.2] states the structure of the generic fibres of the Hitchin map.

Proposition 5.4.2. *Let $s \in \mathcal{H}_0$. If X_s is smooth, then the fibre $H^{-1}(s)$ is isomorphic to*

$$\mathrm{Prym}(X_s/X) = \{L \in \mathrm{Pic}(X_s) : \det \pi_* L \cong \xi\}$$

In the non-strongly parabolic case, in [LM10] it is proved that the fibre over points $s \in \mathcal{H}$ with smooth spectral curves X_s is isomorphic to the Jacobian of the spectral curve, $\mathrm{Jac}(X_s)$.

Chapter 6

Simpson correspondences

In this chapter we will describe the interactions between Higgs bundles and some other related geometric structures. In [Sim08], Simpson studies a set of correspondences between the moduli space of Higgs bunddles over a Riemann surface X, the moduli space of connections over X and the representations of $\pi_1(X)$ in $\mathrm{GL}(r, \mathbb{C})$. In fact, it proves that there exist an isomorphism of the corresponding categories.

In [Sim95], these correspondences are extended for noncompact curves, using the formalism of parabolic vector bundles introduced in chapter 4. The concept of filtered object is introduced and filtered versions of Higgs bundles, connections and representations are presented.

The correspondences for the compact case can be derived from the noncompact one taking an empty set of punctures over the curve. Thus, we will only describe the parabolic version of the objects and theorems.

On the other hand, after stating the general theorem, we will consider certain subclasses of the filtered objects by introducing some extra hypothesis. We will prove that the correspondences hold when restricted to these subclasses. In the language of principal bundles, all the moduli spaces we have treated before correspond to GL-bundles. These new narrower families will be used later on in order to build moduli spaces corresponding to SL-bundles.

6.1 Filtered objects

In this section we will introduce two families of filtered systems, parabolic connections and filtered local systems.

6.1.1 Parabolic Connections

One of the most important notion used in Riemannian geometry is the concept of connection. Let E be a complex vector bundle over X.

Definition 6.1.1. *A holomorphic connection ∇ on E is a $\mathbb{C}$-linear homomorphism of sheaves, $\nabla : E \to E \otimes K$ satisfying the Leibnitz identity, i.e., of f is a locally defined holomorphic function on $\mathcal{O}_X$ and s is a locally defined holomorphic section of E then*

$$\nabla(fs) = f \cdot \nabla(s) + s \otimes df \tag{6.1.1}$$

The moduli space of connections on X over vector bundles E of rank r and degree zero will be denoted by $\mathcal{M}_{conn}(X,r)$. The rank will be dropped from the notation whenever it is clear.

We will now consider connections on a vector bundle that may have logarithmic singularities at some prescribed points of X. Let $D=\{x_1,\dots,x_n\}$ be a set of points on X.

Definition 6.1.2. *A logarithmic connection on a vector bundle E over X with singularities over D is a $\mathbb{C}$-linear homomorphism of sheaves $\nabla : E \to E\otimes K(D)$ satisfying the Leibnitz identity* (6.1.1) *for every locally defined holomorphic function f on $\mathcal{O}_X$ and every locally defined holomorphic section of E, s.*

As we have previously discussed in the case of parabolic Higgs bundles, we can think of logarithmic connections both as homomorphisms of $\nabla : E \to E\otimes K(D)$ or as meromorphic morphisms $\nabla : E \to E\otimes K$, with poles of order at most one on D.

The moduli space of semistable logarithmic connections was built in [Nit93].

Given a logarithmic connection ∇ on a vector bundle E, for every $x\in D$, the Poincaré adjunction formula states that $\mathcal{O}_X(D)|_x \cong T_xX$, so

$$K(D)|_x \cong K|_x \otimes \mathcal{O}_X(D)|_x \cong T_x^*X \cong T_xX \cong \mathbb{C}$$

Then, for every $x\in D$, ∇ induces an endomorphism of every fiber E_x

$$\nabla|_{E_x} : E_x \to E_x \otimes K(D)|_x \cong E_x$$

This endomorphism is called the residue of ∇ at x and it is denoted by $\mathrm{Res}(\nabla,x)$. Intuitively, the residues of a logarithmic connection at the points in D control the way the connection diverges at the punctures. For every $x\in D$, it is natural to consider the eigenvectors and eigenvalues of $\mathrm{Res}(\nabla,x)$, as they indicate both the directions and the rates of divergence of ∇ when it approaches the pole x.

If we assume that the spectrum of $\mathrm{Res}(\nabla,x)$ is real and all the eigenvalues are different real numbers in the interval $[0,1)$, then the eigenvectors of $\mathrm{Res}(\nabla,x)$ induce a filtration of E_x in the following way. Let $\{v_i(x)\}$ be the set of ordered eigenvectors of $\mathrm{Res}(\nabla,x)$, i.e., such that the corresponding eigenvalues $\{\alpha_i(x)\}$ form an increasing sequence. Let us define

$$E_{i,x} = \mathrm{span}(v_1(x),\dots,v_{r-i}(x))$$

Clearly, $E_{i,x}$ form a decreasing full flag filtration of E_x for every $x\in D$. If we consider the corresponding eigenvalues as weights of the filtration, the pair $(E_{i,x},\alpha_i(x))$ form a parabolic structure of E over D. The construction of the spaces $E_{i,x}$ implies that $\mathrm{Res}(\nabla,x)(E_{i,x}) \subseteq E_{i,x}$ for every i and every $x\in D$. Thus

$$\nabla(E_{x,i}) \subseteq E_{x,i}\otimes K(D)|_x$$

for every i and every $x\in D$. Therefore, ∇ is a parabolic morphism of sheaves with respect to the given parabolic structure.

Thus, the parabolic structure of the parabolic vector bundle $(E,E_{i,x})$ reflects the behavior of the connection ∇ at the parabolic points.

In general, the spectrum of $\mathrm{Res}(\nabla,x)$ will not have the previous properties. Nevertheless, following the same ideas, the previous remarks lead up to the following definition of parabolic connection.

Definition 6.1.3. *Let α be a system of weights on D. A parabolic connection on X parabolic over D (for the group $\mathrm{GL}(r, \mathbb{C})$) is a pair (E, ∇) where*

1. *$E \longrightarrow X$ is a parabolic vector bundle of rank r and weight system α.*

2. *$\nabla : E \to E \otimes K(D)$ is a $\mathbb{C}$-linear homomorphism of sheaves over the underlying vector space of E satisfying the following conditions*

   (a) If f is a locally defined holomorphic function on $\mathcal{O}_X$ and s is a locally defined holomorphic section of E then

   $$\nabla(fs) = f \cdot \nabla(s) + s \otimes df$$

   (b) For each $x \in D$ the homomorphism induced in the filtration over the fiber E_x, i.e., the residue, satisfies

   $$\nabla(E_{x,i}) \subseteq E_{x,i} \otimes K(D)|_x$$

If (E, ∇) is a parabolic connection, then $\mathrm{Res}(\nabla, x)$ preserves the filtration on every $x \in D$. Thus, for every $x \in D$ every $i = 1, \dots, r$, $\mathrm{Res}(\nabla, x)$ acts linearly on $E_{x,i}/E_{x,i-1}$. Therefore, the action is the multiplication by $\gamma_i(x)$ for some $\gamma_i(x) \in \mathbb{C}$. As a result, the spectrum of $\mathrm{Res}(\nabla, x)$ is given by $\{\gamma_i(x)\}_i$.

If (E, ∇) is a parabolic connection, the parabolic structure of E essentially defines the residue of ∇ at the punctures. Later on, we will see that, under certain conditions, prescribing a system of weights for the parabolic structure is equivalent to fixing the spectrum of the residue at each point in D.

The category of parabolic connections over a Riemann surface X, together with a set of punctures D is studied in [Sim90].

6.1.2 Filtered local systems

Let $X_{\mathbb{R}}$ be the underlying real manifold of dimension 2 of X. Let $x_0 \in X_{\mathbb{R}}$ be an arbitrary point and let G be a group. A representation of the fundamental group of X is an homomorphism $\pi_1(X_{\mathbb{R}}, x_0) \to G$. The space of representations of the fundamental group of a curve in a given group G is a fundamental tool in algebraic topology.

Connections over vector bundles and representations are related by the Riemann-Hilbert correspondence. Let ∇ be a connection on a vector bundle E over X. Let $\gamma \in \pi_1(X_{\mathbb{R}}, x_0)$. The Riemann-Hilbert correspondence sends γ to the monodromy of ∇ arround γ. This gives a linear automorphism of the fibre E_{x_0}, so it is an element of the group $\mathrm{GL}(r, \mathbb{C})$. It is clear that the composition of two loops is sent to the multiplication of the corresponding monodromies, so we get an homomorphism $\pi_1(X_{\mathbb{R}}, x_0) \to \mathrm{GL}(r, \mathbb{C})$.

Reciprocally, given a representation $\rho : \pi_1(X_{\mathbb{R}}, x_0) \to \mathrm{GL}(r, \mathbb{C})$, we can obtain a vector bundle $E_X(\rho)$ together with a connection $\nabla_X(\rho)$ considering the trivial rank r bundle on the universal covering of $X_{\mathbb{R}}$, say Y. Then $X_{\mathbb{R}} = Y/\pi_1(X_{\mathbb{R}}, x_0)$. The vector bundle $E_X(\rho)$ is defined identifying the fibres of the trivial bundle $Y \times \mathbb{C}^r$ over equivalent points. Let $p, q \in Y$ such that there exist a loop $\gamma \in \pi_1(X_{\mathbb{R}}, x_0)$ such that $\gamma \cdot p = q$. Then for every $v \in \mathbb{C}^r$, we identify the point $(p, v) \in Y \times \mathbb{C}^r$ with $(q, \rho(\gamma)v) \in Y \times \mathbb{C}^r$.

This defines a vector bundle of rank r over X. Under the described identifications, the trivial connection on $Y \otimes \mathbb{C}^r$ induces a connection $\nabla_X(\rho)$ on $E_X(\rho)$.

It is clear that two representations of the fundamental group that differ only by the action of $\mathrm{GL}(r, \mathbb{C})$ by conjugation induce the same vector bundle and connection, as the action simply produces a linear fibrewise isomorphism. Thus, it is natural to consider classes of representations under the equivalence class given by the described action of $\mathrm{GL}(r, \mathbb{C})$. Thus, we will define the moduli space of representations of $X_{\mathbb{R}}$ in $\mathrm{GL}(r, \mathbb{C})$ as

$$\mathcal{M}_{rep}(X_{\mathbb{R}}) = \mathcal{M}_{rep}(X, \mathrm{GL}(r, \mathbb{C})) = \mathrm{Hom}(\pi_1(X_{\mathbb{R}}, x_0), \mathrm{GL}(r, \mathbb{C})) /\!\!/ \mathrm{GL}(r, \mathbb{C})$$

This moduli is also called the Betti moduli space for the curve X and the group $\mathrm{GL}(r, \mathbb{C})$. It was described by Simpson in [Sim94] and [Sim95].

We would like to extend this kind of correspondence for noncompact curves. Thus, we need to extend the notion of representation in order to reflect the structure of a singular connection at the punctured points. The key point of view in order to state the needed equivalence is the one given by local systems.

Definition 6.1.4. *A local system is a vector bundle on X given by constant transition functions.*

A local system is exactly equivalent to giving a vector space L_{x_0}, i.e., the fibre of the local system over x_0, and a representation of $\pi_1(X_{\mathbb{R}}, x_0)$ on L_{x_0}.

Let us suppose that we have a punctured Riemann curve X with punctures on D and a local system L defined over $X \backslash D$. Let $x \in D$ be a puncture. Let us fix once and for all a ray ρ_x emanating out from x. Algebraically, it makes sense to take the stalk of L over ρ_x. We will denote that stalk by L_x. If ρ_x is extended back to x_0, then L_x is identified with L_x.

Let us fix a generator γ_x of $\pi_1(X_{\mathbb{R}}, x_0)$ corresponding to "continuing once around x". Let us consider the monodromy transformation of L_x around γ_x. It will be denoted by μ_x.

Definition 6.1.5. *A filtered local system is a local system L together with left continuous decreasing filtrations $L_{\beta,x}$ of the stalks L_x for every $x \in D$, indexed by real numbers β such that for every $\beta \geq 0$ and $x \in D$*

$$\mu_x(L_{\beta,x}) \subseteq L_{\beta,x}$$

As $L_{\beta,x}$ is a left continuous decreasing filtration, it can be described in terms of a weighted flag over L_x similarly to parabolic vector bundles. Thus, a filtered local system is completely described by taking weighted flags

$$L_x = L_{0,x} \supsetneq L_{1,x} \supsetneq \cdots \supsetneq L_{k,x} \supsetneq \{0\}$$

together with weights

$$0 \leq \beta_1(x) \leq \beta_1(x) \leq \cdots \leq \beta_k(x)$$

In this occasion, we will not ask the filtration to be full flag, as we will see that full flag filtrations on connections may not correspond, in general, to full flags on the corresponding filtered local system.

Similarly to the definition of the parabolic degree given in chapter 4, we can define a parabolic degree function for filtered local systems as follows.

Definition 6.1.6. *Let $(L, L_{\beta,x})$ be a filtered local system. We define the parabolic degree of L as*

$$\operatorname{pardeg}(L) := \sum_{x\in D}\sum_{i\geq 1}\beta_i(x)\dim(L_{i,x}/L_{i-1,x})$$

Alternatively, it can clearly be described in terms of the grading as

$$\operatorname{pardeg}(L) := \sum_{x\in D}\sum_{\beta\geq 0}\beta\dim(\operatorname{Gr}_\beta(\{L_{\beta,x}\})$$

Once we have defined the parabolic degree, we define the parabolic slope of a filtered local system as

$$\operatorname{par}\mu(L) := \frac{\operatorname{pardeg}(L)}{\operatorname{rk}(L)}$$

Therefore, we get a notion of stability of filtered local systems

Definition 6.1.7. *A filtered local system L is (semi)stable if for any subsystem $M \subsetneq L$ with the induced filtration,*

$$\operatorname{par}\mu(M)(\leq) < \operatorname{par}\mu(L) \quad .$$

Here, the notion of subsystem reduces clearly to the notion of parabolic subbundles described in chapter 4, restricting it to bundles with constant transition functions. Thus, all the discussion about why is it sufficient to test the stability condition on parabolic subbundles with the induced parabolic structure hold for subsystems.

6.2 Simpson correspondences

In the previous sections we have described the Riemann-Hilbert correspondence as a bijection between connections and representations. Once we have built the corresponding moduli spaces, we can wonder about the regularity of such bijection with respect to the variety structures of both moduli spaces. The following theorem states this precisely

Theorem 6.2.1 (Riemann-Hilbert correspondence)**.** *The map sending each representation $\rho : \pi_1(X_{\mathbb{R}}) \to \mathrm{GL}(r,\mathbb{C})$ to the pair $(E_X(\rho), \nabla_X(\rho))$ previously defined defines a biholomorphic isomorphism*

$$\mathcal{M}_{rep}(X_{\mathbb{R}}) \xrightarrow{\sim} \mathcal{M}_{conn}(X)$$

Nevertheless, this map is not algebraic, as the monodromy around every loop is computed essentially as an exponential map. The relation between Higgs bundles and local systems is not as explicit as this one and it is due to Simpson. It involves the notion of equivariant harmonic maps and the construction of harmonic metrics over the parabolic vector bundles. A complete description of this relation can be found in [Sim95]. The following theorem states the parabolic version of the correspondence.

Theorem 6.2.2 (Simpson)**.** *There exist a natural one to one correspondence between stable parabolic Higgs bundles of parabolic degree zero, stable filtered local systems of parabolic degree zero or equivalently stable parabolic connections of degree zero.*

Moreover, [Sim90, Theorem 3] implies that the correspondence described in the theorem defines a complete equivalence of the categories of parabolic Higgs bundles, parabolic connections and filtered local systems. Additionally, it can be proved that the maps induced between the corresponding moduli spaces are diffeomorphisms.

Proposition 6.2.3. *The moduli space of stable parabolic Higgs bundles of zero parabolic degree, the moduli space of stable parabolic connections of zero parabolic degree and the moduli space of stable filtered local systems of parabolic degree zero are pairwise diffeomorphic.*

We will now focus on the correspondences existing between the parabolic structures of all three filtered objects. The details of the correspondences between filtrations is described in [Sim90]. We will only discuss the transformations of the parabolic weights and the corresponding eigenvalues of the residues at the punctures.

Let (E,Φ), (V,∇) and L be a parabolic Higgs bundle, a parabolic connection and a filtered local system, all three mutually related by the correspondence given in theorem 6.2.2. Let us fix a point $x \in D$ and fix a particular weight α for the underlying parabolic bundle of (E,Φ). As Φ preserves the filtration, the residue of the Higgs bundle at x given by the restriction

$$\Phi|_{E_x} : E_x \to E_x \otimes K(D)|_x \cong E_x$$

has an associated eigenvalue for each element of the filtration. Let $b + ci$ be the eigenvalue corresponding to the element of the filtration of weight α. We can study how the weight and the corresponding eigenvalue of the residue are changed by the correspondence on each case.

The permutations of the parabolic weights and the eigenvalues of the residues are given by this table [Sim95, page 720]

	(E,Φ)	(V,∇)	L
weight	α	$\alpha - 2b$	$\beta = -2b$
eigenvalue	$b + ci$	$\alpha + 2ci$	$\exp(-2\pi i\alpha + 4\pi c)$

(6.2.1)

6.3 Further hypothesis

In order to simplify the structure of the considered moduli spaces, we will assume some additional hypothesis on the structure of parabolic Higgs fields and parabolic connections.

First of all, as we said in chapter 3, we will only consider strictly parabolic Higgs bundles. Thus, for each $x \in D$, the residue of every parabolic Higgs bundle (E, Φ) at x, $\mathrm{Res}(\Phi, x)$, will have zero spectrum. Thus, in the table (6.2.1) we are fixing $b = c = 0$.

Therefore, substituting $b = c = 0$ in (6.2.1), we obtain a simplified description of the permutation of the weights and eigenvalues given by the following table

	(E, Φ)	(V, ∇)	L
weight	α	α	$\beta = 0$
eigenvalue	0	α	$\exp(-2\pi i\alpha)$

(6.3.1)

Observing the previous table we can deduce two principal consequences of considering strictly parabolic Higgs bundles. The first one is that the corresponding parabolic connection shares the same system of weights of E. Moreover, by fixing the system of weights, we are also fixing the spectrum of the residue of the connection ∇.

Taking into account this properties, we will use the following restricted definition of a parabolic connection.

Definition 6.3.1. *Let ξ be a fixed vector bundle and let α be a system of weights on D. A parabolic connection on X parabolic over D (for the group* $\mathrm{GL}(r, \mathbb{C})$*) is a pair (E, ∇) where*

1. *$E \longrightarrow X$ is a parabolic vector bundle of rank r and weight system α.*
2. *$\nabla : E \to E \otimes K(D)$ is a $\mathbb{C}$-linear homomorphism of sheaves over the underlying vector space of E satisfying the following conditions*
   - *(a) If f is a locally defined holomorphic function on $\mathcal{O}_X$ and s is a locally defined holomorphic section of E then*
   $$\nabla(fs) = f \cdot \nabla(s) + s \otimes df$$
   - *(b) For each $x \in D$ the homomorphism induced in the filtration over the fiber E_x satisfies*
   $$\nabla(E_{x,i}) \subseteq E_{x,i} \otimes K(D)|_x$$
   - *(c) For every $x \in D$ and every $i = 1, \ldots, r$ the action of* $\mathrm{Res}(\nabla, x)$ *on $E_{x,i}/E_{x,i-1}$ is the multiplication by $\alpha_i(x)$. Since* $\mathrm{Res}(\nabla, x)$ *preserves the filtration, it acts on each quotient.*

Furthermore, similarly to Higgs bundles, we will consider parabolic connections whose underlying parabolic vector bundle has a prescribed determinant. This leads up to the following definition. Let ξ be a line bundle over X and let α be a system of weights on D such that $\sum_{i=1}^{r} \alpha_i(x) \in \mathbb{Z}$ for every $x \in D$. Let us consider the parabolic line bundle (ξ, ξ_β) over ξ defined taking trivial filtrations over each $x \in D$ with parabolic weights

$$\beta(x) = \sum_{i=1}^{r} \alpha_i(x)$$

This parabolic line bundle has an associated parabolic Higgs bundle taking the zero Higgs field. Let $(\xi, \nabla_{\xi,\alpha})$ be the parabolic connection corresponding to the parabolic Higgs bundle $(\xi, 0)$ through the Simpson correspondence.

Definition 6.3.2. *Let ξ be a fixed vector bundle and let α be a system of weights on D. A parabolic connection on X parabolic over D (for the group* $\mathrm{SL}(r,\mathbb{C})$*) is a pair (E,∇) where*

1. *$E \longrightarrow X$ is a* SL*-parabolic vector bundle of rank r and weight system α together with an isomorphism $\bigwedge^r E \cong \xi$*
2. *$\nabla : E \to E \otimes K(D)$ is a $\mathbb{C}$-linear homomorphism of sheaves over the underlying vector space of E satisfying the following conditions*
   (a) *If f is a locally defined holomorphic function on $\mathcal{O}_X$ and s is a locally defined holomorphic section of E then*
   $$\nabla(fs) = f \cdot \nabla(s) + s \otimes df$$
   (b) *For each $x \in D$ the homomorphism induced in the filtration over the fiber E_x satisfies*
   $$\nabla(E_{x,i}) \subseteq E_{x,i} \otimes K(D)|_x$$
   (c) *For every $x \in D$ and every $i = 1, \ldots, r$ the action of* $\mathrm{Res}(\nabla, x)$ *on $E_{x,i}/E_{x,i-1}$ is the multiplication by $\alpha_i(x)$. Since* $\mathrm{Res}(\nabla, x)$ *preserves the filtration, it acts on each quotient.*
   (d) *The operator $\bigwedge^r E \to \bigwedge^r E \otimes K(D)$ induced by ∇ coincides with $\nabla_{\xi,\alpha}$.*

Given a Riemann surface X, together with a set of punctures D, a fixed system of weights α over D and a line bundle ξ, there exist a moduli space of parabolic connections over X, parabolic over D whose underlying parabolic vector bundle is semistable of rank r, determinant ξ and system of weights α. We will denote this moduli space by $\mathcal{M}_{conn}(r, \alpha, \xi, X)$ and we will omit the curve X whenever it can be deduced from the context. If X has genus $g \geq 2$, this moduli space is a complex algebraic variety of dimension

$$\dim(\mathcal{M}_{conn}(r, \alpha, \xi)) = 2(g-1)(r^2-1) + n(r^2-r) = \dim(\mathcal{M}_{Higgs}(r, \alpha, \xi))$$

The notion of parabolic connection for the group $\mathrm{SL}(r,\mathbb{C})$ will be studied more deeply in chapter nine.

The second consequence of considering strictly parabolic Higgs bundles is that the filtration of the local system is constantly zero. This implies that the filtration of a local system corresponding to a strictly parabolic Higgs bundle is trivial.

As the filtration is exactly the same for all the representations corresponding to strictly parabolic Higgs bundles, we can drop the filtration from the structure and consider simply local systems, i.e., representations of the fundamental group in the usual sense.

Therefore, the Simpson correspondence, when restricted to strictly parabolic Higgs fields yields,

Theorem 6.3.3. *There exist a natural one to one correspondence between stable strictly parabolic Higgs bundles of parabolic degree zero,* $\mathrm{GL}(r, \mathbb{C})$*-representations of the fundamental group and stable parabolic connections of degree zero in the sense of 6.3.1.*

Chapter 7

Torelli theorems

Through chapters §2 to §5 we have introduced different kind of moduli spaces consisting on schemes whose underlying set of closed points corresponds to the set of equivalence classes of fibre spaces of a certain kind over a fixed projective complex algebraic curve X.

It is clear that these moduli spaces depend heavily on the algebraic structure of the base curve X. For example, the Jacobian variety of a sphere and the Jacobian variety over a torus must be different simply because of topological reasons. If X and Y are isomorphic it is clear that $\mathrm{Jac}(X) \cong \mathrm{Jac}(Y)$, but it is not obvious whether the Jacobian variety contains sufficiently enough information of the base curve in order to state the converse.

Torelli theorem is a classical result in the theory of algebraic curves that precisely states this reciprocal whenever the Jacobian is presented as a principally polarized variety.

Definition 7.0.1. *A polarization of a scheme $\mathcal{M}$ is an ample line bundle λ over $\mathcal{M}$. The pair $(\mathcal{M}, \lambda)$ is called a polarized variety. A morphism of polarized varieties $f : (\mathcal{M}, \lambda) \to (\mathcal{M}', \lambda')$ is a morphism $f : \mathcal{M} \to \mathcal{M}'$ such that the pullback of λ' by f is λ.*

The category of polarized schemes is clearly well defined and so, the concept of isomorphism of polarized schemes is natural. For every curve X, the Jacobian $\mathrm{Jac}(X)$ can be given a canonical polarization. The details of this construction can be found in [CS86, §7.6]. Whereas there does not exist a Torelli type theorem for the Jacobian itself, the polarization induced on the Jacobian by the curve has enough information about the latter in order to allow the following theorem.

Theorem 7.0.2 (Torelli's theorem)**.** *Let X and X' be complete smooth curves of genus $g \geq 2$ over an algebraically closed field k. Let λ and λ' be the canonical polarizations of $\mathrm{Jac}(X)$ and $\mathrm{Jac}(X')$ respectively. If $(\mathrm{Jac}(X), \lambda)$ is isomorphic to $(\mathrm{Jac}(X'), \lambda')$ as polarized varieties over k then X and X' are isomorphic over k.*

In the literature, this is known as the classical Torelli theorem. A proof of the theorem can be found in [CS86, Corollary §7.12.2] taking into account that every algebraically closed field is a perfect field.

The essence of Torelli type theorems such as 7.0.2 is that we have a certain moduli space $\mathcal{M}(X)$ parameterizing the classes of a certain geometric object over a curve X (vector bundles over X, Higgs bundles over X, etc.). Then the isomorphism class of the moduli space $\mathcal{M}(X)$, maybe with some additional geometric structure, uniquely determines the isomorphism class of the curve X.

After treating the line bundle scenario, the natural generalization would be that of general vector bundles over a complex smooth projective curve X. As we described in Chapter §2, there is no suitable moduli space for the set of classes of all possible vector bundles over X, so we will restrict ourselves to the moduli space of semistable vector bundles over X. In [MN68], Mumford and Newstead proved the following Torelli theorem for the moduli space of stable rank 2 and fixed odd degree determinant.

Theorem 7.0.3. *Let X and X' be complex smooth curves of genus $g \geq 2$. Let ξ and ξ' be line bundles of odd degree over X and X' respectively. Let $\mathcal{M}^s(2,\xi,X)$ and $\mathcal{M}^s(2,\xi',X')$ be the moduli spaces of stable vector bundles of rank 2 and determinant ξ or ξ' respectively. If $\mathcal{M}^s(2,\xi,X)$ is isomorphic to $\mathcal{M}^s(2,\xi',X')$ then X is isomorphic to X'.*

This result was generalized by Narasimhan and Ramanan in [NR75], where they extended it for any rank whenever the rank r and the degree of ξ are coprime.

Theorem 7.0.4. *Let X and X' be complex smooth curves or genus $g \geq 2$. Let $\xi \to X$ and $\xi' \to X'$ be line bundles of degree d and $r \geq 2$ be coprime to d. If $\mathcal{M}(r,\xi,X)$ is isomorphic to $\mathcal{M}(r,\xi',X')$ then X is isomorphic to X'.*

In order to do so, they manage to polarize an intermediate Jacobian (generalization of the Jacobian for schemes of dimension higher than one) of $\mathcal{M}(r,\xi,X)$ so that it is isomorphic to $\mathrm{Jac}(X)$ as a polarized variety. This way, they reduce the proof to the classical Torelli theorem. A further generalization of this theorem for arbitrary rank and degree was prooved by Kouvidakis and Pantev in [KP95].

Continuing with the previously described moduli spaces, the next scenario is that of finding a Torelli type theorem for the moduli space of Higgs bundles over a curve. Biswas and Gómez proved such a theorem in [BG03]

Theorem 7.0.5. *Let X and X' be complex smooth curves of genus $g \geq 2$. Let ξ and ξ' be line bundles on X and X' respectively such that $\deg(\xi) = \deg(\xi') = d$. Let r be an integer coprime with d. If $\mathcal{M}_{Higgs}(r,\xi,X)$ is isomorphic to $\mathcal{M}_{Higgs}(r,\xi',X')$ as an algebraic variety then X and X' are isomorphic.*

The key in the proof of this theorem is that, as we stated in (3.2.1), the moduli space of semistable vector bundles over X, $\mathcal{M}(r,\xi,X)$ can be embedded into $\mathcal{M}_{Higgs}(r,\xi,X)$ taking the Higgs field as constantly zero. Thus, it is enough to identify the corresponding subvariety into the abstract variety $\mathcal{M}_{Higgs}(r,\xi,X)$ in an intrinsic way, i.e., in a way that only depends of the structure of $\mathcal{M}_{Higgs}(r,\xi,X)$ as an abstract variety.

We recall, that in all this Torelli type theorems, we are only given the moduli space as an abstract variety, so a priori we have no additional information about its structure. For example, in the case of $\mathcal{M}_{Higgs}(r,\xi,X)$, we can't use directly the information given by the Hitchin map (3.3.4), because the Hitchin map is only one of many morphisms from the abstract variety $\mathcal{M}_{Higgs}(r,\xi,X)$ to the Hitchin space.

If we want to use the information provided by this map explicitly, we have to find an intrinsic way of characterizing it from the set of all such possible morphisms.

Similarly, the scheme $\mathcal{M}(r,\xi,X)$ is just one of many subschemes of $\mathcal{M}_{Higgs}(r,\xi,X)$. The proof of the theorem is based on characterizing it uniquely as the irreducible component of the fix point set for all the $\mathbb{C}^*$ actions on $\mathcal{M}_{Higgs}(r,\xi,X)$ of maximal dimension.

This will be a general strategy for many of the Torelli type theorems that we will study. We will build up a"stair" of Torelli theorems on which each theorem is proved localizing a subscheme into the moduli space for which we have a Torelli theorem.

Once we have stated some of the main Torelli theorems for smooth curves that we will use, we will study some of the correspondent theorems in the parabolic scenario.

In [BdBnB01], Balaji, del Baño and Biswas proved a Torelli theorem for parabolic vector bundles of rank 2 and "small" parabolic weights analogous to that of Mumford and Newstead described in theorem 7.0.3.

Theorem 7.0.6. *Let S (respectively S') be a finite subset of a compact connected Riemann surface X (respectively X') of genus $g \geq 2$. Let $\mathcal{M}$ (respectively $\mathcal{M}'$) denote the moduli space of parabolic stable bundles of rank two over X (respectively X') with fixed determinant of degree one, having a nontrivial parabolic structure over each point of S (respectively, S'), and of parabolic degree less than two. Then $\mathcal{M}$ is isomorphic to $\mathcal{M}'$ if and only if there is an isomorphism of X with X' taking S to S'.*

The condition of "small" parabolic weights is taken so that parabolic stability is equivalent to regular stability of the underlying vector bundle. Later on we will see that we can relax this condition allowing "big" parabolic weight whenever they are "concentrated" enough in a certain way. Nowadays, there does not exist a generalization of this theorem for arbitrary rank.

In [GL11], Gómez and Logares proved a version of theorem 7.0.5 for the parabolic scenario.

Theorem 7.0.7. *Let X and X' be smooth projective curves of genus $g \geq 2$ with marked points $D = \{x_1, \dots, x_n\} \subset X$ and $D' = \{x'_1, \dots, x'_n\} \subset X'$, let $\mathcal{M}_{Higgs}(2,\alpha,\xi,X)$ and $\mathcal{M}_{Higgs}(2,\alpha,\xi',X')$ the moduli spaces of parabolic Higgs bundles over X and X' respectively, with fixed determinant ξ and ξ' of odd degree, and the same small weights α (over D or D' respectively). If there is an isomorphism between $\mathcal{M}_{Higgs}(2,\alpha,\xi,X)$ and $\mathcal{M}_{Higgs}(2,\alpha,\xi',X')$ then there is an isomorphism between X and X' that sends D to D'.*

The main idea of the proof is similar to that of the Torelli theorem for Higgs bundles over a smooth curve [BG03]. As we saw in (5.3.1), the moduli space $\mathcal{M}(r,\alpha,\xi,X)$ can be embedded in $\mathcal{M}_{Higgs}(r,\alpha,\xi,X)$ fixing the Higgs field as zero. They show that $\mathcal{M}(r,\alpha,\xi,X)$ is the only irreducible component of the nilpotent cone which does not admit a nontrivial $\mathbb{C}^*$-action. Once the moduli space of parabolic vector bundles has been characterized, they apply 7.0.6 to retrieve the corresponding curves and divisors.

An important remark about the proof of this theorem is that the hypothesis of rank 2 bundles is only necessary because of the usage of theorem 7.0.6. If the theorem 7.0.6 was extended for higher rank then theorem 7.0.7 would also hold with the same proof under the hypothesis of small weights, full flags and coprimality of rank and degree.

Chapter 8

Deligne-Hitchin moduli space

The Simpson correspondences described in chapter 6 state that there exist isomorphisms between the moduli spaces of Higgs bundles, the moduli of connections and the moduli of representations of the fundamental group of the underlying real manifold. In this chapter we will work in the category of SL-vector bundles. We recall that in order to acquire this, we will impose the following conditions.

- All vector bundles have trivial determinant $\det(E) \cong \mathcal{O}_X$.
- Higgs bundles are meant to have a traceless Higgs field.
- We will consider SL-representations, i.e., representations of the fundamental group in $\mathrm{SL}(r, \mathbb{C})$.

It can be proved that $\mathcal{M}_{Higgs}(r, \mathcal{O}_X, X)$ and $\mathcal{M}_{conn}(r, \mathcal{O}_X, X)$ are not isomorphic as complex varieties. The Riemann-Hilbert map induces a biholomorphism between $\mathcal{M}_{rep}(r, X_{\mathbb{R}})$ and $\mathcal{M}_{Conn}(r, \mathcal{O}_X, X)$. On the other side, all three varieties are isomorphic as real manifolds. Thus, these spaces induce two different complex structures, on the underlying manifold of $\mathcal{M}_{Higgs}(r, \mathcal{O}_X, X)$.

Let us call I and J the complex structures induced by $\mathcal{M}_{Higgs}(r, \mathcal{O}_X, X)$ and $\mathcal{M}_{conn}(r, \mathcal{O}_X, X)$ respectively. It can be proved that $\mathcal{M}_{Higgs}(r, \mathcal{O}_X, X)$ can be given a structure of Hyperkahler manifold, with complex structures I, J and $K = IJ$.

The Deligne-Hitchin moduli space can be constructed as the twistor space associated to $\mathcal{M}_{Higgs}(r, \mathcal{O}_X, X)$. Particularly, this space gives a way of gluing together the complex varieties with complex structures I and J, corresponding to the spaces $\mathcal{M}_{Higgs}(r, \mathcal{O}_X, X)$ and $\mathcal{M}_{conn}(r, \mathcal{O}_X, X)$.

In this chapter we will describe the Deligne-Hitchin moduli space using a different approach. Firstly, we will define the Hodge moduli space of a Riemann surface, $\mathcal{M}_{Hod}(r, X)$. This space will parametrize λ-connections on vector bundles. λ-connections generalize both Higgs bundles and connections, so this space will "stick together" the spaces $\mathcal{M}_{Higgs}(r, \mathcal{O}_X, X)$ and $\mathcal{M}_{conn}(r, \mathcal{O}_X, X)$. Then, if we take $\overline{X}$ to be the conjugate curve of X, we will define the Deligne-Hitchin moduli space identifying the general fibre of $\mathcal{M}_{Hod}(r, X)$ and $\mathcal{M}_{Hod}(r, \overline{X})$ using the Riemann-Hilbert correspondence.

8.1 Hodge moduli space

Higgs bundles and connections over a vector bundle have some similar properties. Let (E, Φ) be a Higgs field and let (E, ∇) be a connection. Then

- Both of them are defined over an underlying vector bundle E over X.
- The Higgs field Φ and the connection ∇ are both $\mathbb{C}$-linear homomorhisms of sheaves $E \to E \otimes K$
- If f is a locally defined holomorphic function on SO_X and s is a locally defined holomorphic section of E, then

$$\begin{cases} \Phi(fs) = f\Phi(s) \\ \nabla(fs) = f\nabla(s) + s \otimes df \end{cases}$$

λ-connections, defined in the sense of [Sim94, page 87] and [Sim08, page 4], generalize Higgs bundles and connections at the same time.

Definition 8.1.1. *Given any $\lambda \in \mathbb{C}$, a λ-connection on X for the group* $\mathrm{SL}(r, \mathbb{C})$ *is a pair (E, ∇), where*

a) E is a holomorphic SL*-vector bundle of rank r together with an isomorphism* $\det(E) \cong \mathcal{O}_X$.

b) $\nabla : E \to E \otimes K$ is a $\mathbb{C}$-linear homomorphism of sheaves satisfying the following two conditions

1) If f is a locally defined holomorphic function on SO_X and s is a locally defined holomorphic section of E, then

$$\nabla(fs) = f\nabla(s) + \lambda \cdot s \otimes df$$

2) The operator $\bigwedge^r E \to \bigwedge^r E \otimes K$ induced by ∇ coincides with $\lambda \cdot d$

If $\lambda = 0$, condition (1) implies that the 0-connection $\nabla : E \to E \otimes K$ is a linear homomorphism of $\mathcal{O}_X$-sheaves. By (2), the induced morhism $\nabla : \bigwedge^r E \to \bigwedge^r E \otimes K$ is the zero morphism. As this morphism coincides with the trace of ∇, ∇ is traceless, so it is an element of $\mathcal{M}_{Higgs}(r, \mathcal{O}_X, X)$.

If $\lambda = 1$, a 1-connection $\nabla : E \to E \otimes K$ is a $\mathbb{C}$-linear homomorphism such that $\nabla(fs) = f\nabla(s) + s \otimes df$ for very locally defined holomorphic function f on $\mathcal{O}_X$ and every locally defined holomorphic section of E, s. Thus, ∇ is a SL-connection in the usual sense.

We consider the moduli space of triples of the form (λ, E, ∇), where λ is a complex number and (E, ∇) is a λ-connection on X for the group $\mathrm{SL}(r, \mathbb{C})$. [Del89] defines precisely the corresponding moduli problem and proves that there exist a corresponding coarse moduli space. This space is called the Hodge moduli space for the group $\mathrm{SL}(r, \mathbb{C})$ on X, and will be denoted by

$$\mathcal{M}_{Hod}(r, X) = \mathcal{M}_{Hod}(X, \mathrm{SL}(r, \mathbb{C}))$$

If X is a Riemann surface of genus $g \geq 2$, the moduli space is a complex algebraic variety of dimension $1 + 2(r^2 - r)(g-1)$. It is naturally equipped with a surjective algebraic morphism

$$\mathrm{pr}_\lambda : \mathcal{M}_{Hod}(r, X) \to \mathbb{C}$$

given by $(\lambda, E, \nabla) \mapsto \lambda$. We know that 0-connections correspond to Higgs bundles, so we get an embedding

$$\mathcal{M}_{Higgs}(r, \mathcal{O}_X, X) = \mathrm{pr}_\lambda^{-1}(0) \hookrightarrow \mathcal{M}_{Hod}(r, X)$$

In particular, as $\mathcal{M}(r, \mathcal{O}_X, X)$ is embedded into $\mathcal{M}_{Higgs}(r, \mathcal{O}_X, X)$, it is also embedded into $\mathcal{M}_{Hod}(r, X)$. On the other hand, 1-connections are connections in the usual sense, so

$$\mathcal{M}_{conn}(r, \mathcal{O}_X, X) = \mathrm{pr}_\lambda^{-1}(1) \hookrightarrow \mathcal{M}_{Hod}(r, X)$$

The rest of the fibers of pr_λ are isomorphic to $\mathrm{pr}_\lambda^{-1}(1)$. To proof this, let us consider a λ-connection ∇ over a vector bundle E for some $\lambda \neq 0$. Then condition (1) yields

$$\nabla(fs) = f\nabla(s) + \lambda \cdot s \otimes df$$

Dividing both sides of the equation by ∇ we obtain

$$\frac{1}{\lambda}\nabla(fs) = f\frac{1}{\lambda}\nabla(s) + s \otimes df$$

Thus, $\frac{1}{\lambda}\nabla$ is a 1-connection. Using this property, $\mathbb{C}^*$ acts on $\mathcal{M}_{Hod}(r, X)$ as

$$t \cdot (\lambda, E, \nabla) \mapsto (t \cdot \lambda, E, t \cdot \nabla)$$

The set of fixed points of this action lies inside the zero fibre, i.e., inside the moduli space of semistable Higgs bundles. [BGHL09] use this fact, together with the analysis of the $\mathbb{C}^*$ actions on $\mathcal{M}_{Higgs}(r, \mathcal{O}_X, X)$ in order to characterize the immersion of $\mathcal{M}(r, \mathcal{O}_X, X)$ into $\mathcal{M}_{Hod}(r, X)$ uniquely. The subvariety is determined using the following lemma.

Lemma 8.1.2. *Let Z be an irreducible component of the subspace of $\mathcal{M}_{Hod}(r, X)$ that is fixed for all the $\mathbb{C}^*$ actions on $\mathcal{M}_{Hod}(r, X)$. Then $\dim(Z) \leq (r^2 - 1)(g - 1)$, with equality only for $Z = i(\mathcal{M}(r, \mathcal{O}_X, X))$.*

The previous lemma allows us to identify the subvariety $\mathcal{M}(r, \mathcal{O}_X, X)$ inside $\mathcal{M}_{Hod}(r, X)$ intrinsically. Thus, if $\mathcal{M}_{Hod}(r, X) \cong \mathcal{M}_{Hod}(r, Y)$ for some other Riemann surface Y of the same genus, then $\mathcal{M}(r, \mathcal{O}_X, X) \cong \mathcal{M}(r, \mathcal{O}_Y, Y)$. Therefore, applying the Torelli theorem for the moduli space of semistable vector bundles, we obtain a Torelli theorem for $\mathcal{M}_{Hod}(r, X)$.

Theorem 8.1.3 (Torelli theorem for the Hodge moduli space)**.** *The isomorphism class of the complex analytic space $\mathcal{M}_{Hod}(r, X)$ determines uniquely the isomorphism class of the Riemann surface X.*

8.2 Deligne-Hitchin space

We will introduce Deligne's construction [Del89] of the Deligne-Hitchin moduli space as described in [Sim08, page 7]. Let $X_{\mathbb{R}}$ be the underlying real C^∞ manifold of X. We recall that the moduli space of SL-representations of the fundamental group of $X_{\mathbb{R}}$ is defined as

$$\mathcal{M}_{rep}(X_{\mathbb{R}}) = \mathrm{Hom}(\pi_1(X_{\mathbb{R}}, x_0), \mathrm{SL}(r,\mathbb{C})) /\!\!/ \mathrm{SL}(r,\mathbb{C})$$

where x_0 is any fixed point $x_0 \in X_{\mathbb{R}}$.

The Riemann-Hilbert correspondence described in chapter 6 defines a biholomorphic isomorphism $\mathcal{M}_{rep}(X_{\mathbb{R}}) \xrightarrow{\sim} \mathcal{M}_{conn}(r, \mathcal{O}_X, X)$ sending each representation $\rho \in \mathcal{M}_{rep}(X_{\mathbb{R}})$ to the pair $(E_X(\rho), \nabla_X(\rho))$. As every nonzero fibre of $\mathcal{M}_{Hod}(r,X)$ is isomorphic to $\mathcal{M}_{conn}(r, \mathcal{O}_X, X)$, for every $\lambda \in \mathbb{C}^*$, we can associate a representation $\rho \in \mathcal{M}_{rep}(X_{\mathbb{R}})$ the λ-connection $(\lambda, E_X(\rho), \lambda \cdot \nabla_X(\rho))$. This defines an holomorphic open embedding

$$\mathbb{C}^* \times \mathcal{M}_{rep}(X_{\mathbb{R}}) \longrightarrow \mathcal{M}_{Hod}(r, X) \tag{8.2.1}$$

onto the open locus $\mathrm{pr}_\lambda^{-1}(\mathbb{C}^*)$ of all triples (λ, E, ∇) with $\lambda \neq 0$.

Let J_X be the almost complex structure of X defined on $X_{\mathbb{R}}$. Then $-J_X$ is also an almost complex structure on $X_{\mathbb{R}}$. Let $\overline{X}$ be the Riemann surface defined taking the structure $-J_X$ on $X_{\mathbb{R}}$.

A priori, the isomorphism classes of the Higgs moduli space of X and $\overline{X}$ are different. Nevertheless, as X and $\overline{X}$ share the same underlying real manifold $X_{\mathbb{R}}$, the moduli spaces of SL-representations of X and $\overline{X}$ are the same. As the moduli spaces of connections, $\mathcal{M}_{conn}(r, \mathcal{O}_X, X)$ and $\mathcal{M}_{conn}(r, \mathcal{O}_{\overline{X}}, \overline{X})$ on X and $\overline{X}$ respectively are both isomorphic to $\mathcal{M}_{rep}(X_{\mathbb{R}})$ as complex varieties, we have natural isomorphisms between the nonzero fibres of $\mathcal{M}_{Hod}(r,X)$ and $\mathcal{M}_{Hod}(r,\overline{X})$. We will use this map to build the Deligne-Hitchin moduli space.

Definition 8.2.1. *Let X be a Riemann surface. The Deligne-Hitchin moduli space for X, $\mathcal{M}_{DH}(r,X)$ is defined as*

$$\mathcal{M}_{DH}(r,X) := \mathcal{M}_{Hod}(r,X) \cup \mathcal{M}_{Hod}(r,\overline{X})$$

by gluing together $\mathcal{M}_{Hod}(r,X)$ and $\mathcal{M}_{Hod}(r,\overline{X})$ along the image of $\mathbb{C}^* \times \mathcal{M}_{rep}(X_{\mathbb{R}})$ by the map (8.2.1)*. For each $\lambda \in \mathbb{C}^*$ and each $\rho \in \mathcal{M}_{rep}(X_{\mathbb{R}})$, we identify the points*

$$(\lambda, E_X(\rho), \lambda \cdot \nabla_X(\rho)) \in \mathcal{M}_{Hod}(r,X) \quad \textit{and} \quad (\lambda^{-1}, E_{\overline{X}}(\rho), \lambda^{-1} \cdot \nabla_{\overline{X}}(\rho)) \in \mathcal{M}_{Hod}(r,\overline{X})$$

Deligne proves in [Del89] that the space $\mathcal{M}_{DH}(r,X)$ is a complex analytic space of dimension $2(r^2-1)(g-1)+1$. The space lacks a natural algebraic structure because the Riemann-Hilbert correspondence used to glue $\mathcal{M}_{Hod}(r,X)$ and $\mathcal{M}_{Hod}(r,\overline{X})$ is only holomorphic and not algebraic.

The map $\mathrm{pr}_\lambda : \mathcal{M}_{Hod}(r,X) \to \mathbb{C}$ extends naturally to a holomorphic morphism

$$\mathrm{pr} : \mathcal{M}_{DH}(r,X) \to \mathbb{CP}^1$$

Taking into account all the previously described relations between the fibres of the Hodge space and the Riemann-Hilbert correspondence, we can describe each of the fibers of pr. Let $\lambda \in \mathbb{CP}^1$. The fibre over λ is canonically isomorphic (as a complex variety) to

- $\mathcal{M}_{Higgs}(r, \mathcal{O}_X, X)$ if $\lambda = 0$.
- $\mathcal{M}_{Higgs}(r, \mathcal{O}_{\overline{X}}, \overline{X})$ if $\lambda = \infty$.
- $\mathcal{M}_{conn}(r, \mathcal{O}_X, X)$ if $\lambda \neq 0$ and $\lambda \neq \infty$. Using the Riemann-Hilbert correspondence, this fibres are also isomorphic to $\mathcal{M}_{rep}(X_{\mathbb{R}})$.

As we have seen in the previous section, there exist a natural $\mathbb{C}^*$ action on both Hodge moduli spaces, $\mathcal{M}_{Hod}(r, X)$ and $\mathcal{M}_{Hod}(r, \overline{X})$. Let us see that we can extend this natural action from $\mathcal{M}_{Hod}(r, X)$ to $\mathcal{M}_{DH}(r, X)$. Let us consider the action on $\mathcal{M}_{Hod}(r, \overline{X})$. Under the gluing, the action of every $t \in \mathbb{C}^*$ over the image of $\mathbb{C}^* \times \mathcal{M}_{rep}(X_{\mathbb{R}}) \to \mathcal{M}_{Hod}(r, X)$ coincides to the action of t^{-1} on the image of $\mathbb{C}^* \times \mathcal{M}_{rep}(X_{\mathbb{R}}) \to \mathcal{M}_{Hod}(r, \overline{X})$. Therefore, we obtain an action of $\mathbb{C}^*$ on $\mathcal{M}_{DH}(r, X)$.

It is important to notice that the Deligne-Hitchin moduli space of X is exactly the same as the one of $\overline{X}$. Therefore, we can't expect to have a Torelli theorem for this moduli space which allows us to recover the original curve X. Nevertheless, we can prove that we can describe uniquely the unordered pair of Riemann surfaces $\{X, \overline{X}\}$.

Theorem 8.2.2 (Torelli theorem for the Deligne-Hitchin moduli space)**.** *The isomorphism class of the complex analytic space $\mathcal{M}_{DH}(r, X)$ determines uniquely the isomorphism class of the unordered pair of Riemann surfaces $\{X, \overline{X}\}$.*

The proof of the theorem is given in [BGHL09, Theorem 4.1]. It is based on the proof given for the Torelli theorem for the Hodge moduli space. We have both $\mathcal{M}(r, \mathcal{O}_X, X)$ and $\mathcal{M}(r, \mathcal{O}_{\overline{X}}, \overline{X})$ embedded into $\mathcal{M}_{DH}(r, X)$. We can characterize both subspaces as the irreducible components of maximum dimension inside the fixed point locus of $\mathcal{M}_{DH}(r, X)$ with respect to all the $\mathbb{C}^*$ actions.

There will exist exactly two irreducible components. The first one will be embedded into $\mathcal{M}_{Higgs}(r, \mathcal{O}_X, X) = \mathrm{pr}^{-1}(0)$ and will be given by the immersion of $\mathcal{M}(r, \mathcal{O}_X, X)$. The other one will be embedded into $\mathcal{M}_{Higgs}(r, \mathcal{O}_{\overline{X}}, \overline{X}) = \mathrm{pr}^{-1}(\infty)$ and will be given by the immersion of $\mathcal{M}(r, \mathcal{O}_{\overline{X}}, \overline{X})$.

Thus, the isomorphism class of $\mathcal{M}_{DH}(r, X)$ determines the isomorphism class of the unordered pair of moduli spaces $\{\mathcal{M}(r, \mathcal{O}_X, X), \mathcal{M}(r, \mathcal{O}_{\overline{X}}, \overline{X})\}$. Therefore, using [KP95, page 229, Theorem E] we conclude that $\mathcal{M}_{DH}(r, X)$ determines the isomorphism class of the unordered pair of Riemann surfaces $\{X, \overline{X}\}$.

Chapter 9

Torelli theorem for the parabolic Deligne-Hitchin moduli space

In this chapter we present a generalization of the Torelli theorem for the Deligne-Hitchin moduli space of a compact curve given in [BGHL09]. We will use the formalism of parabolic vector bundles in order to extend this theorem to punctured Riemann surfaces.

Firstly, we will prove some additional propositions about parabolic vector bundles and parabolic Higgs bundles. Using them, we will be able to give an alternative proof to the Torelli theorem for parabolic Higgs bundles. This proof is different from the one provided by [GL11] and adapts the techniques used in [BGHL09] to the parabolic case.

We will consider a parabolic version of the Hodge moduli space for a punctured Riemann surface. Similarly to the compact case, a Torelli theorem for the parabolic Hodge moduli space will be proven.

Finally, we will use a parabolic version of the Riemann-Hilbert correspondence to build a parabolic analogue of the Deligne-Hitchin moduli space. Similarly to the compact case, this space will be the twistor space of the moduli space of parabolic Higgs bundles. The main result of this work is a Torelli theorem for this space.

9.1 Parabolic Vector Bundles

Let X be a smooth projective curve over $\mathbb{C}$ of genus $g \geq 2$. Let D be a finite set of $n \geq 1$ distinct points of X. We recall (remark given after definition 4.1.2) that a parabolic vector bundle over X is a holomorphic vector bundle of rank r together with a weighted full flag on the fiber E_x over each $x \in D$ called parabolic structure, i.e.

$$E_x = E_{x,0} \supsetneq E_{x,1} \supsetneq \cdots \supsetneq E_{x,r} = \{0\}$$

$$0 \leq \alpha_1(x) < \cdots < \alpha_r(x) < 1$$

We denote $\alpha = \{(\alpha_1(x), \dots, \alpha_r(x))\}_{x\in D}$ to the system of real weights corresponding to a fixed parabolic structure.

Let ξ be a line bundle over X. Let $\mathcal{M}(r, \alpha, \xi, X)$ be the moduli space of semi-stable parabolic vector bundles on X of rank r with weight system α together with an isomorphism $\bigwedge^r E \cong \xi$. We will omit the curve X whenever it is clear. As we explained in chapter 4, it is a projective scheme of dimension

$$\dim(\mathcal{M}(r, \alpha, \xi)) = (g-1)(r^2-1) + \frac{n(r^2-r)}{2}$$

Let $\mathcal{M}^s(r, \alpha, \xi)$ be the open subvariety parameterizing the parabolically stable bundles. This open subvariety lies inside the smooth locus of $\mathcal{M}(r, \alpha, \xi)$.

Proposition 9.1.1. *Fix an integer $r \geq 2$. Then for a generic system of weights $\alpha = \{\alpha_1(x), \dots, \alpha_r(x)\}_{x\in D}$, every parabolically semi-stable vector bundle E over X with* $\operatorname{pardeg}(E) = 0$ *is parabolically stable.*

Proof. For each $I \subsetneq \{1, \dots, r\}$, $I \neq \emptyset$ and $m \in \{0, \dots, nr\}$ let $A_{I,m} = \{\alpha : \sum_{i\in I, x\in D} \alpha_i(x) = m\}$. Let $A = \bigcup_{I\subsetneq\{1,\dots,r\}, I\neq\emptyset} \bigcup_{m=0}^{nr} A_{I,m}$. For every I, m, $A_{I,m}$ is clearly closed. As A is a union of a finite number of closed sets, A is closed. Let's prove that for all $\alpha \notin A$, every parabolically semi-stable vector bundle with respect to α is parabolically stable. Let $\alpha \notin A$ and let E be a parabolically semi-stable vector bundle over X with respect to α with $\operatorname{pardeg}(E) = 0$. Then for all $x \in D$

$$\deg(E) + \sum_{x\in D} \sum_{i=1}^{r} \alpha_i(x) = \operatorname{pardeg}(E) = 0$$

Suppose than E is strictly parabolically semi-stable. Then there exists a subbundle F such that

$$\frac{\operatorname{pardeg}(F)}{\operatorname{rk}(F)} = \frac{\operatorname{pardeg}(E)}{\operatorname{rk}(E)} = 0$$

The parabolic structure in F is the one induce by E, so for all $x \in D$, its weight system is a proper subset $I_F(x)$ of $\operatorname{rk}(F) < r$ elements of $\alpha_1(x), \dots, \alpha_r(x)$. Then we have

$$0 = \operatorname{pardeg}(F) = \deg(F) + \sum_{x\in D} \sum_{i\in I_F(x)} \alpha_i(x)$$

As $0 \leq \alpha_i(x) < 1$, we have $0 \leq \sum_{x\in D} \sum_{i\in I_F(x)} \alpha_i(x) < nr$. As $\deg(F) \in \mathbb{Z}$, we have $\sum_{x\in D} \sum_{i\in I_F(x)} \alpha_i(x) = -\deg(F) \in \{0, \dots, nr-1\}$, so $\alpha \in A_{I_F(x), -\deg(F)}(x)$, which contradicts $\alpha \in A$. $\square$

We have an analogous proof if $deg(E)$ is fixed. In particular, given a fixed determinant ξ of degree $-d < 0$, the condition $\operatorname{pardeg}(E) = 0$ in the previous proposition is equivalent to $\sum_{x\in X} \sum_{i\in I} \alpha_i(x) = d$ and we get that for a generic choice of the system weight α among those with $\sum_{x\in X} \sum_{i\in I} \alpha_i(x) = d$ every parabolically semi-stable vector bundle is parabolically stable.

Corollary 9.1.2. *Let ξ be a vector bundle of negative degree. For a generic system of weights α such that $\sum_{x\in X}\sum_{i\in I}\alpha_i(x) = -\deg(\xi)$*

$$\mathcal{M}(r,\alpha,\xi) = \mathcal{M}^s(r,\alpha,\xi)$$

We will now study the relation between stability and parabolical stability for a certain type of weight systems.

Definition 9.1.3. *Fix a rank r. A system of weights $\alpha = \{(\alpha_1(x),\dots,\alpha_r(x))\}_{x\in D}$ is said to be concentrated if $\alpha_r(x) - \alpha_1(x) < \frac{1}{n(r-1)^2}$ for all $x \in D$.*

Lemma 9.1.4. *Let α be a concentrated system of weights. Let $I = \{1,\dots,r\}$. Then for all $x \in D$ and for all $I'(x) \subsetneq I$, $I'(x) \neq \emptyset$, $|I'(x)| = r'$*

$$\left|\sum_{x\in D}\left(r\sum_{i\in I'(x)}\alpha_i(x) - r'\sum_{i\in I}\alpha_i(x)\right)\right| < 1$$

Proof.

$$\left|\sum_{x\in D}\left(r\sum_{i\in I'(x)}\alpha_i(x) - r'\sum_{i\in I}\alpha_i(x)\right)\right| = \left|\sum_{x\in D}\sum_{i\in I'(x)}\sum_{j\in I}(\alpha_i(x)-\alpha_j(x))\right| \leq$$

$$\leq \sum_{x\in D}\sum_{i\in I'(x)}\sum_{j\in I}|\alpha_i(x)-\alpha_j(x)| = \sum_{x\in D}\sum_{i\in I'(x)}\sum_{j\in I\setminus I'(x)}|\alpha_i(x)-\alpha_j(x)| <$$

$$< \sum_{x\in D}\sum_{i\in I'(x)}\sum_{j\in I\setminus I'(x)}\frac{1}{n(r-1)^2} = \frac{n(r-r')r'}{n(r-1)^2} \leq 1$$

□

Proposition 9.1.5. *Let α be a concentrated system of weights for rank r. Then every vector bundle E over X with $\gcd(\deg(E),\mathrm{rk}(E)) = 1$ the following are equivalent*

1. *E is semi-stable*
2. *E is stable*
3. *E is parabolically semi-stable with respect to α*
4. *E is parabolically stable with respect to α*

Proof. We will prove (1) $\Rightarrow$ (2) $\Rightarrow$ (4) $\Rightarrow$ (3) $\Rightarrow$ (1). By definition of parabolic (semi)-stability (4) $\Rightarrow$ (3).

(1) $\Rightarrow$ (2) Let E be a parabolic vector bundle over X. E is semi-stable iff for every subbundle $\{0\} \neq F \neq E$ we have

$$\frac{\deg(F)}{\mathrm{rk}(F)} \leq \frac{\deg(E)}{\mathrm{rk}(E)} \quad .$$

Or, equivalently, iff

$$\deg(F)\operatorname{rk}(E) - \deg(E)\operatorname{rk}(F) \leq 0 \quad .$$

Let us suppose that E is strictly semi-stable. Then there exist a subbundle F with $0 \neq F \neq E$ such that

$$\deg(F)\operatorname{rk}(E) - \deg(E)\operatorname{rk}(F) = 0 \quad .$$

So $\deg(F)\operatorname{rk}(E) = \deg(E)\operatorname{rk}(F)$ and we have $\operatorname{rk}(E)|\deg(E)\operatorname{rk}(F)$. As $\gcd(\deg(E), \operatorname{rk}(E)) = 1$ we must have $\operatorname{rk}(E)|\operatorname{rk}(F)$. Nevertheless, F is a subbundle of E with $\{0\} \neq F \neq E$, so $0 < \operatorname{rk}(F) < \operatorname{rk}(E)$ and we arrive to a contradiction.

(2) $\Rightarrow$ (4) For every subboundle F we consider the system of weights α_F induced by α on F. Then for every $x \in D$ there exist a subset $I_F(x) \subsetneq I$ with $|I_F(x)| = \operatorname{rk}(F)$ such that $\alpha_F = \{\alpha_i : i \in I_F(x)\}_{x\in D}$.

E is parabolically stable with respect to α iff for every subbundle $\{0\} \neq F \neq E$ and for every $x \in D$

$$\frac{\deg(F) + \sum_{x\in D}\sum_{i\in I_F(x)} \alpha_i(x)}{\operatorname{rk}(F)} < \frac{\deg(E) + \sum_{x\in D}\sum_{i\in I} \alpha_i(x)}{\operatorname{rk}(E)}$$

or equivalently, iff

$$\deg(F)\operatorname{rk}(E) - \deg(E)\operatorname{rk}(F) < \sum_{x\in D}\left(\operatorname{rk}(E)\sum_{i\in I_F(x)}\alpha_i(x) - \operatorname{rk}(F)\sum_{i\in I}\alpha_i(x)\right) \quad .$$

On the other hand if E is stable then for every subbundle $0 \neq F \neq E$

$$\deg(F)\operatorname{rk}(E) - \deg(E)\operatorname{rk}(F) < 0 \quad .$$

As $\deg(F)\operatorname{rk}(E) - \deg(E)\operatorname{rk}(F) \in \mathbb{Z}$ we have $\deg(F)\operatorname{rk}(E) - \deg(E)\operatorname{rk}(F) \leq -1$. Lemma 9.1.4 implies

$$-1 < \sum_{x\in D}\left(\operatorname{rk}(E)\sum_{i\in I_F(x)}\alpha_i(x) - \operatorname{rk}(F)\sum_{i\in I}\alpha_i(x)\right) < 1$$

so for every subbundle $\{0\} \neq F \neq E$

$$\deg(F)\operatorname{rk}(E) - \deg(E)\operatorname{rk}(F) \leq -1 < \sum_{x\in D}\left(\operatorname{rk}(E)\sum_{i\in I_F(x)}\alpha_i(x) - \operatorname{rk}(F)\sum_{i\in I}\alpha_i(x)\right)$$

(3) $\Rightarrow$ (1) If E is parabolically semi-stable with respect to α then for every subbundle $\{0\} \neq F \neq E$

$$\deg(F)\operatorname{rk}(E) - \deg(E)\operatorname{rk}(F) \leq \sum_{x\in D}\left(\operatorname{rk}(E)\sum_{i\in I_F(x)}\alpha_i(x) - \operatorname{rk}(F)\sum_{i\in I}\alpha_i(x)\right) \quad .$$

As a consequence of lemma 9.1.4,

$$\sum_{x\in D}\left(\mathrm{rk}(E)\sum_{i\in I_F(x)}\alpha_i(x)-\mathrm{rk}(F)\sum_{i\in I}\alpha_i(x)\right)<1 \quad .$$

Therefore,

$$\deg(F)\,\mathrm{rk}(E)-\deg(E)\,\mathrm{rk}(F)\leq\sum_{x\in D}\left(\mathrm{rk}(E)\sum_{i\in I_F(x)}\alpha_i(x)-\mathrm{rk}(F)\sum_{i\in I}\alpha_i(x)\right)<1.$$

$\deg(F)\,\mathrm{rk}(E)-\deg(E)\,\mathrm{rk}(F)\in\mathbb{Z}$ and $\deg(F)\,\mathrm{rk}(E)-\deg(E)\,\mathrm{rk}(F)<1$, so for all subbundle $0\neq F\neq E$, $\deg(F)\,\mathrm{rk}(E)-\deg(E)\,\mathrm{rk}(F)\leq 0$ and so, E is semistable. □

From now on we will suppose that ξ has negative degree $-d$ and α is a concentrated system of weights with $\sum_{x\in D}\sum_{i\in I}\alpha_i(x)=d$. The weights will be required to be concentrated in order to apply later on the Torelli theorem for the moduli space of rank 2 semistable parabolic vector bundles given in [BdBnB01]. Nevertheless, if there existed a generalization of the Torelli theorem in [BdBnB01] for generic parabolic weights, the proofs given in this chapter would also hold for generic parabolic weights.

Lemma 9.1.6. *The holomorphic cotangent bundle*

$$T^*\mathcal{M}^s(r,\alpha,\xi)\longrightarrow\mathcal{M}^s(r,\alpha,\xi)$$

does not admit any nonzero holomorphic section.

Proof. As a consequence of proposition 9.1.5, $\mathcal{M}^s(r,\alpha,\xi)=\mathcal{M}(r,\alpha,\xi)$. Then $\mathcal{M}(r,\alpha,\xi)$ is smooth. As we are considering full flags on every point in D, by [BY99, theorem 6.1] $\mathcal{M}(r,\alpha,\xi)$ is rational. Then it is a smooth rational projective variety, so it does not admit any nonzero holomorphic 1-form. □

9.2 Parabolic Higgs bundles

As we will only work with strictly parabolic Higgs bundles, we will call them simply Higgs bundles. We will use "non-strictly parabolic Higgs bundles" otherwise. Moreover, we will consider $\mathrm{SL}(r,\mathbb{C})$-Higgs bundles with a prescribed determinant ξ, where ξ is a line bundle over X. As we said in chapter 3, this implies that we will only consider traceless Higgs bundles.The reference to the group $\mathrm{SL}(r,\mathbb{C})$ will be omitted from this point.

We denote by $\mathcal{M}_{Higgs}(r,\alpha,\xi)$ the moduli space of parabolically semi-stable (strictly parabolic) Higgs bundles of rank r and weight system α and $\mathrm{tr}\,\Phi=0$ together with an isomorphism $\bigwedge^r E\cong\xi$. In chapter 5 we stated that it is an irreducible normal projective variety of dimension

$$\dim(\mathcal{M}_{Higgs}(r,\alpha,\xi))=2(g-1)(r^2-1)+n(r^2-r)$$

We will also call $\mathcal{M}^{sm}_{Higgs}(r,\alpha,\xi)$ the smooth locus of $\mathcal{M}_{Higgs}(r,\alpha,\xi)$. Finally, let $\mathcal{M}^{ns}_{Higgs}(r,\alpha)$ be the moduli space of non-strictly parabolic Higgs bundles of rank r and weight system α.

There is a natural embedding

$$i : \mathcal{M}(r,\alpha,\xi) \hookrightarrow \mathcal{M}_{Higgs}(r,\alpha,\xi) \tag{9.2.1}$$

defined by $E \mapsto (E,0)$. Let $\mathcal{M}^{st}_{Higgs}(r,\alpha,\xi)$ be the locus of Higgs bundles (E,Φ) whose underlying vector bundle E is parabolically stable. It is an open dense subset of $\mathcal{M}_{Higgs}(r,\alpha,\xi)$. Let

$$\mathrm{pr}_E : \mathcal{M}^{st}_{Higgs}(r,\alpha,\xi) \longrightarrow \mathcal{M}^s(r,\alpha,\xi) \tag{9.2.2}$$

be the forgetful map defined by $(E,\Phi) \to E$. By deformation theory, the tangent space at $[E]$, $T_{[E]}\mathcal{M}^s(r,\alpha,\xi)$ is isomorphic to $H^1(X, \mathrm{ParEnd}(E))$. By the parabolic version of Serre duality,

$$H^1(X, \mathrm{ParEnd}(E))^* \cong H^0(X, \mathrm{SParEnd}(E) \otimes K(D))$$

and hence, the Higgs field is an element of the cotangent bundle $T^*_{[E]}\mathcal{M}^s(r,\alpha,\xi)$ and one has a canonical isomorphism

$$\mathcal{M}^{st}_{Higgs}(r,\alpha,\xi) \xrightarrow{\sim} T^*\mathcal{M}^s(r,\alpha,\xi) \tag{9.2.3}$$

of varieties over $\mathcal{M}^s(r,\alpha,\xi)$.

Proposition 9.1.5 implies that $\mathcal{M}^s(r,\alpha,\xi) = \mathcal{M}(r,\alpha,\xi)$ so we get an isomorphism

$$\mathcal{M}^{st}_{Higgs}(r,\alpha,\xi) \xrightarrow{\sim} T^*\mathcal{M}(r,\alpha,\xi) \tag{9.2.4}$$

Let us recall the definition of the Hitchin map and the Hitchin space for non-strictly parabolic Higgs bundles. Let $S = \mathbb{V}(K(D))$ be the total space of the line bundle K(D), let

$$p : S = \underline{\mathrm{Spec}}\, \mathrm{Sym}^\bullet(K^{-1} \otimes \xi(D)^{-1}) \longrightarrow X$$

be the projection, and $x \in H^0(S, p^*(K(D)))$ be the tautological section. The characteristic polynomial of a Higgs field

$$\det(x \cdot \mathrm{id} - p^*\Phi) = x^r + \tilde{s_1}x^{r-1} + \tilde{s_2}x^{r-2} + \cdots + \tilde{s_r}$$

defines sections $s_i \in H^0(X, K^iD^i)$, such that $\tilde{s_i} = p^*s_i$ and K^iD^j denotes the tensor product of the i-th power of K with the j-th power of the line bundle associated to D. We define the Hitchin space as

$$\mathcal{H} = \bigoplus_{i=1}^{r} H^0(K^iD^i) \tag{9.2.5}$$

The Hitchin map is defined as

$$H^{ns} : \mathcal{M}^{ns}_{Higgs}(r,\alpha) \longrightarrow \mathcal{H} \tag{9.2.6}$$

sending each Higgs bundle (E,Φ) to the characteristic polynomial of Φ.

We can now restrict the Hitchin map to $\mathcal{M}_{Higgs}(r,\alpha,\xi)$. We are assuming that Φ is strongly parabolic, therefore the residue at each point of D is nilpotent. This implies that the eigenvalues of Φ vanish at D, so for each $i > 0$ the section s_i belongs to the subspace $H^0(X, K^iD^{i-1}) \subseteq H^0(X, K^iD^i)$. Moreover, in order to fix the determinant, we are asking Φ to be traceless, so $s_1 = 0$ and the image in the Hitchin space lies in

$$\mathcal{H}_0 = \bigoplus_{i=2}^{r} H^0(K^iD^{i-1}) \tag{9.2.7}$$

Therefore, one obtains a map

$$H : \mathcal{M}_{Higgs}(r,\alpha,\xi) \longrightarrow \mathcal{H}_0 \tag{9.2.8}$$

Lemma 9.2.1. *The Hitchin map restricted to $\mathcal{M}_{Higgs}(r,\alpha,\xi)$, (9.2.8) is projective.*

Proof. By [Yok93, Corollary 5.12] the map (9.2.6) is projective. We clearly have an immersion

$$j : \mathcal{M}_{Higgs}(r,\alpha,\xi) \longrightarrow \mathcal{M}^{ns}_{Higgs}(r,\alpha) \quad .$$

$\mathcal{M}_{Higgs}(r,\alpha,\xi)$ is the subset of $\mathcal{M}^{ns}_{Higgs}(r,\alpha)$ of Higgs bundles (E,Φ), such that $det(E) \cong \xi$, $\operatorname{tr}\Phi = 0$ and Φ is strictly parabolic. The condition of being strictly parabolic can be locally set imposing that for every $x \in D$ and every choice of local coordinates for the bundle around x coherent with the given filtration, Φ has zeros in the diagonal entries. Therefore, all three conditions are closed and the map j is a closed map.

We now have the following commutative diagram

$$\begin{tikzcd}
\mathcal{M}_{Higgs}(r,\alpha,\xi) \arrow[r, "j"] \arrow[d, "H"] & \mathcal{M}^{ns}_{Higgs}(r,\alpha) \arrow[r, "f"] \arrow[d, "H'"] & \mathbb{P}^n_{\mathcal{H}} \arrow[dl] \\
\mathcal{H}_0 \arrow[r] & \mathcal{H} &
\end{tikzcd}$$

As H' is projective, it factors into a closed immersion $f : \mathcal{M}^{ns}_{Higgs}(r,\alpha) \to \mathbb{P}^n_{\mathcal{H}}$ for some n, followed by the projection $\mathbb{P}^n_{\mathcal{H}} \to \mathcal{H}$. Then $H' \circ j$ factors into another closed immersion $f \circ j : \mathcal{M}_{Higgs}(r,\alpha,\xi) \to \mathbb{P}^n_{\mathcal{H}}$ and the projection $\mathbb{P}^n_{\mathcal{H}} \to \mathcal{H}$, so it's projective. As H is just the restriction of the image of $j \circ H'$ to $\mathcal{H}_0$, H must be projective. □

Lemma 9.2.2. *The fibers of the Hitchin map (9.2.8) are connected.*

Proof. By [GL11, Lemma 3.1] and [GL11, Lemma 3.2], the fibers of (9.2.8) over a certain open dense subset U of $\mathcal{H}_0$ are isomorphic to a Prym variety, so each of those fibers are connected. Applying Stein factorization theorem [Har10, Corollary 11.5] to the projective morphism H gives us an algebraic variety $\tilde{\mathcal{H}}_0$ and morphisms $\tilde{H}$ and g such that $\tilde{H}$ has connected fibers, g is a finite morphism and the following diagram commutes

$$\begin{tikzcd} \mathcal{M}_{Higgs}(r,\alpha,\xi) \arrow[r, "\tilde{H}"] \arrow[dr, "H"'] & \tilde{\mathcal{H}}_0 \arrow[d, "g"] \\ & \mathcal{H}_0 \end{tikzcd}$$

For every $p \in U$, $H^{-1}(p)$ is connected. The image of a connected set is connected, so $\tilde{H}(H^{-1}(p)) = g^{-1}(p)$ is connected. As g is finite, $g^{-1}(p)$ must be a single point. Then g is an isomorphism between $g^{-1}(U)$ and U, so g is a birational map. Every finite morphism is projective, so by By Zariski's Main Theorem [Har10, Corollary 11.4] a birational finite morphism to a normal variety is an isomorphism to an open set, so g is an isomorphism to its image. Thus, every fiber of $H = \tilde{H} \circ g$ is a fiber of $\tilde{H}$ and must be connected.

□

The multiplicative group $\mathbb{C}^*$ acts on the moduli space $\mathcal{M}_{Higgs}(r,\alpha,\xi)$ by

$$t \cdot (E, \Phi) = (E, t\Phi) \tag{9.2.9}$$

The Hitchin map H induces an associated action in $\mathcal{H}$ given by

$$t \cdot (v_2, \dots, v_i, \dots, v_r) = (t^2 v_2, \dots, t^i v_i, \dots, t^r v_r) \tag{9.2.10}$$

Where $v_i \in H^0(X, K^i D^{i-1})$ for $i \in 2, \dots, r$.

Lemma 9.2.3. *The holomorphic tangent bundle*

$$T\mathcal{M}^s(r,\alpha,\xi) \longrightarrow \mathcal{M}^s(r,\alpha,\xi)$$

does not admit any nonzero holomorphic section.

Proof. A holomorphic section s of $T\mathcal{M}^s(r,\alpha,\xi)$ provides by contraction a holomorphic function

$$s^\sharp : T^*\mathcal{M}^s(r,\alpha,\xi) \longrightarrow \mathbb{C}$$

on the total space of the cotangent bundle which is linear on the fibers. Under the isomorphism in (9.2.4), it corresponds to a holomorphic function

$$f : \mathcal{M}^{st}_{Higgs}(r,\alpha,\xi) \longrightarrow \mathbb{C}$$

Taking $\mathcal{G} = SL(\mathcal{O}_X^{\oplus(r-1)} \oplus \xi)$ in [Fal93, Lemma II.6] and [Fal93, V.(iii), page 561] we obtain that the codimension of $\mathcal{M}^{st}_{Higgs}(r,\alpha,\xi)$ in $\mathcal{M}_{Higgs}(r,\alpha,\xi)$ is grater than two.

As $\mathcal{M}_{Higgs}(r,\alpha,\xi)$ is smooth, by Hartog's theorem, the function f extends to a holomorphic function

$$\tilde{f} : \mathcal{M}_{Higgs}(r,\alpha,\xi) \longrightarrow \mathbb{C}$$

Since f is linear on the fibers we know that $\tilde{f}$ must be homogeneous of degree 1 for the action 9.2.9 of $\mathbb{C}^*$. On the moduli space $\mathcal{M}_{Higgs}(r,\alpha,\xi)$ the Hitchin map (9.2.8) is projective by lemma (9.2.1), so it's proper, and its fibers are connected by lemma (9.2.2). Therefore, the function $\tilde{f}$ is constant on the fibers of the Hitchin map and $\tilde{f}$ comes from a holomorfic function on the Hitchin space, which must be still homogeneous of degree 1.

On the other hand, there is no nonzero holomorphic function on $\mathcal{H}$, because all the exponents of t in (9.2.10) are at least two. Therefore, $\tilde{f}=0$ and we get $f=0$, $s^\sharp=0$ and, finally, $s=0$. □

Corollary 9.2.4. *The restriction of the holomorphic tangent bundle*

$$T\mathcal{M}^{sm}_{Higgs}(r,\alpha,\xi) \longrightarrow \mathcal{M}^{sm}_{Higgs}(r,\alpha,\xi)$$

to $i(\mathcal{M}^s(r,\alpha,\xi)) \subseteq \mathcal{M}^{sm}_{Higgs}(r,\alpha,\xi)$ *does not admit any nonzero holomorphic section.*

Proof. Using Lemma 9.2.3, it suffices to show that the normal bundle of the embedding

$$i : \mathcal{M}^s(r,\alpha,\xi) \longrightarrow \mathcal{M}^{st}_{Higgs}(r,\alpha,\xi)$$

does not admit any nonzero holomorphic sections. The isomorphism in (9.2.4) allows us to identify this normal bundle with $T^*\mathcal{M}^s(r,\alpha,\xi)$, so the lemma follows from Lemma 9.1.6. □

We can adapt Simpson's result [Sim95, Lemma 11.9] to the parabolic situation and we obtain the following

Lemma 9.2.5. *Let* (E,Φ) *be a parabolic Higgs bundle in the nilpotent cone, with* $\Phi \neq 0$. *Assume that* (E,Φ) *is a fixed point of the action 9.2.9. Then there is another Higgs bundle* (E',Φ') *in the nilpotent cone, not isomorphic to* (E,Φ) *such that* $\lim_{t\to\infty}(E',t\Phi')=(E,\Phi)$

The previous results combine in

Proposition 9.2.6. *Let* Z *be an irreducible component of the fixed point locus of the action (9.2.9) in* $\mathcal{M}_{Higgs}(r,\alpha,\xi)$. *Then*

$$\dim(Z) \leq (r^2-1)(g-1) + \frac{n(r^2-r)}{2}$$

with equality only for $Z = i(\mathcal{M}^s(r,\alpha,\xi))$.

Proof. The $\mathbb{C}^*$ action 9.2.9 on $\mathcal{M}_{Higgs}(r,\alpha,\xi)$ and the $\mathbb{C}^*$ action 9.2.10 on $\mathcal{H}_0$ are intertwined by the Hitchin map H. Clearly the only fixed point in $\mathcal{H}_0$ for this action is 0, so $Z \subseteq H^{-1}(0)$.

The dimension of $\mathcal{H}_0$ is given by

$$\dim(\mathcal{H}_0) = \sum_{i=2}^{r} \dim\left(H^0(K^iD^{i-1})\right) \tag{9.2.11}$$

Applying Serre duality, the Riemann-Roch theorem yields

$$\dim\left(H^0(K^iD^{i-1})\right) - \dim\left(H^0(K^{1-i}D^{1-i})\right) = \deg(K^iD^{i-1}) - g + 1 \tag{9.2.12}$$

For $i \geq 2$, $\deg(K^iD^{i-1}) = i(2g-2) + (i-1)n$ so $\deg(K^{1-i}D^{1-i}) < 0$, so $\dim\left(H^0(K^{1-i}D^{1-i})\right) = 0$ and we obtain

$$\dim\left(H^0(K^iD^{i-1})\right) = i(2g-2) + (i-1)n - g + 1 \tag{9.2.13}$$

Substituting the computed dimension in equation (9.2.11) yields

$$\begin{aligned} \dim(\mathcal{H}_0) = \sum_{i=2}^{r} (i(2g-2) + (i-1)n - g + 1) = \\ \frac{(r+2)(r-1)}{2}(2g-2) + \frac{r(r-1)}{2}n - (r-1)(g-1) = \\ (r^2-1)(g-1) + \frac{n(r^2-r)}{2} = \dim(\mathcal{M}(r,\alpha,\xi)) \end{aligned} \tag{9.2.14}$$

The fiber $H^{-1}(0)$ is a Lagrangian subscheme of $\mathcal{M}_{Higgs}(r,\alpha,\xi)$ so $\dim(H^{-1}(0)) = \frac{1}{2}\dim(\mathcal{M}_{Higgs}(r,\alpha,\xi)) = \dim(\mathcal{M}(r,\alpha,\xi)$ and so

$$\dim(Z) \leq \dim(\mathcal{M}(r,\alpha,\xi)) = (r^2-1)(g-1) + \frac{n(r^2-r)}{2}$$

and equality holds if Z is an irreducible component of $H^{-1}(0)$.

Recall that $i : \mathcal{M}(r,\alpha,\xi) \to H^{-1}(0)$ takes $E \to (E,0)$. Since a non-trivial $\mathbb{C}^*$-action produces a non-trivial vector field, from Lemma 9.2.3 we know that $\mathcal{M}(r,\alpha,\xi)$ does not admit any non-trivial $\mathbb{C}^*$-action. As $\dim(\mathcal{M}(r,\alpha,\xi)) = \dim(H^{-1}(0))$, $i(\mathcal{M}(r,\alpha,\xi))$ is an irreducible component of $H^{-1}(0)$ of the maximum allowed dimension, so it remains to check that there is no other connected component where there is no $\mathbb{C}^*$-action. The rest of the components have a nonzero-Higgs field, so the $\mathbb{C}^*$- action (9.2.9) $(E,\Phi) \mapsto (E, t\Phi)$ is non-trivial due to Lemma 9.2.5. □

Using the previous Proposition we can obtain a proof of the Torelli theorem for the parabolic Higgs bundles moduli space of a curve

Corollary 9.2.7. *Let $r = 2$. The isomorphism class of the complex analytic space $\mathcal{M}_{Higgs}(r,\alpha,\xi)$ determines uniquely the isomorphism class of the punctured Riemann surface (X,D), meaning that if $\mathcal{M}_{Higgs}(r,\alpha,\xi,X)$ is biholomorphic to $\mathcal{M}_{Higgs}(r,\alpha,\xi',Y)$ for another punctured connected Riemann surface (Y,D') of the same genus g, then $(X,D) \cong (Y,D')$.*

Proof. Let $Z \subset \mathcal{M}_{Higgs}(r,\alpha,\xi)$ be a closed analytic subset with the following three properties:

1. Z is irreducible and has complex dimension $(r^2-1)(g-1) + \frac{n(r^2-r)}{2}$.

2. The smooth locus $Z^{sm} \subseteq Z$ lies in the smooth locus $\mathcal{M}^{sm}_{Higgs}(r,\alpha,\xi) \subset \mathcal{M}_{Higgs}(r,\alpha,\xi)$

3. The restriction of the holomorphic tangent bundle $T\mathcal{M}^{sm}_{Higgs}(r,\alpha,\xi)$ to the subspace $Z^{sm} \subset \mathcal{M}^{sm}_{Higgs}(r,\alpha,\xi)$ has no nonzero holomorphic sections.

By Corollary 9.2.4, the image $i(\mathcal{M}(r,\alpha,\xi))$ of the embedding i in (9.2.1) has these properties. We will prove that this is the only possible choice for Z.

Every $\mathbb{C}^*$ action on $\mathcal{M}_{Higgs}(r,\alpha,\xi)$ defines a holomorphic vector field on $\mathcal{M}^{sm}_{Higgs}(r,\alpha,\xi)$. The third assumption on Z says that any holomorphic vector field on $\mathcal{M}^{sm}_{Higgs}(r,\alpha,\xi)$ vanishes on Z^{sm}. Therefore, the stabilizer of each point $Z^{sm} \subset \mathcal{M}^{sm}_{Higgs}(r,\alpha,\xi)$ has nontrivial tangent space at $1 \in \mathbb{C}^*$, and hence the stabilizer must be the full group $\mathbb{C}^*$.

Then Z^{sm} belongs to the fixed point locus of the action (9.2.9) in $\mathcal{M}_{Higgs}(r,\alpha,\xi)$, and thus, so does its closure in $\mathcal{M}_{Higgs}(r,\alpha,\xi)$, Z. Due to Proposition 9.2.6, and property (1), $Z = i(\mathcal{M}(r,\alpha,\xi))$. In particular, we have $Z \cong \mathcal{M}(r,\alpha,\xi)$.

Then, the isomorphism class of $\mathcal{M}_{Higgs}(r,\alpha,\xi)$ determines the isomorphism class of $\mathcal{M}(r,\alpha,\xi)$. Due to [BdBnB01, Theorem 3.2], the latter determines the isomorphism class of the punctured Riemann surface (X,D). □

The rank two condition of the previous corollary is only necessary in order to apply the Torelli theorem in [BdBnB01]. If the theorem of [BdBnB01] were extended to Higher rank, then corollary 9.2.7 would also hold for higher rank with the same proof given above.

9.3 The parabolic λ-connections

Let ξ be a line bundle over X and let α be a fixed system of weights over D. Let us suppose that $\deg(\xi) = -\sum_{x\in D}\sum_{i=1}^r \alpha_i(x)$. Fixing a line bundle and a system of weights α over X allows us to describe canonically a parabolic line bundle over X, (ξ,ξ_β), taking the underlying vector bundle as ξ and defining trivial filtrations over each $x \in D$ with parabolic weight

$$\beta(x) := \beta_1(x) = \sum_{i=1}^r \alpha_i(x)$$

As ξ has rank one, any parabolic structure on ξ consists of trivial filtrations. Thus, the value of the jump $\beta(x)$ completely defines the parabolic structure on ξ. By construction, we get that

$$\mathrm{pardeg}(\xi) = \deg(\xi) + \sum_{x\in D}\beta(x) = \deg(\xi) + \sum_{x\in D}\sum_{i=1}^r \alpha_i(x) = 0$$

The line bundle ξ can be given the structure of a parabolic Higgs bundle canonically taking a zero Higgs field. In fact, as the rank of ξ is one, every traceless Higgs field over ξ must be zero, so $\mathcal{M}_{Higgs}(1,\beta,\xi)$ consists exactly of the point $(\xi,0)$.

Let (E,Φ) be a traceless strictly parabolic $\mathrm{SL}(r,\mathbb{C})$-Higgs bundle with parabolic system of weights α such that $\det(E) = \xi$. Taking the r-th exterior power, the morphism Φ induces a morphism $\bigwedge^r E \to \bigwedge^r E \otimes K(D)$. It can be proved that the morphism is locally given by the trace of Φ. As $\mathrm{tr}(\Phi) = 0$, the induced morphism is the zero morphism.

Thus, taking the determinant, every parabolic Higgs bundles $[(E,\Phi)] \in \mathcal{M}_{Higgs}(r,\alpha,\xi)$ induces the same parabolic Higgs bundle $(\xi, 0)$.

Using the Simpson correspondence [Sim90] between parabolic Higgs bundles of parabolic degree 0 and parabolic connections of parabolic degree 0, the parabolic Higgs bundle $(\xi, 0)$ corresponds to a parabolic connection $(\xi, \nabla_{\xi,\alpha})$ with the same parabolic weights β, such that $\mathrm{Res}(\nabla_\xi, x) = \beta(x)\,\mathrm{Id}$ for every $x \in D$.

Let (E', ∇) be the parabolic connection corresponding to the Higgs bundle (E,Φ) under the Simpson correspondence. Taking the r-th exterior power, ∇ induces a morphism

$$\tilde{\nabla} : \bigwedge^r E \to \bigwedge^r E \otimes K(D) \quad .$$

The r-th exterior power can be built categorically both in the category of traceless Higgs bundles and in the category of parabolic connections. As the Simpson correspondence is an equivalence of categories, the wedge product of (E', ∇) must be the image of the wedge product of (E, Φ). Therefore, the morphism $\tilde{\nabla}$ must coincide with ∇_ξ. This leads up to the following definition of parabolic λ-connection for the group $\mathrm{SL}(r, \mathbb{C})$.

Definition 9.3.1. *For a fixed line bundle ξ, a system of weights α and a given $\lambda \in \mathbb{C}$ a parabolic λ-connection on X (for the group $\mathrm{SL}(r,\mathbb{C})$) is a pair (E, ∇) where*

1. *$E \longrightarrow X$ is a parabolic vector bundle of rank r and weight system α together with an isomorphism $\bigwedge^r E \cong \xi$*
2. *$\nabla : E \to E \otimes K(D)$ is a $\mathbb{C}$-linear homomorphism of sheaves over the underlying vector space of E satisfying the following conditions*
   - *(a) If f is a locally defined holomorphic function on ξ and s is a locally defined holomorphic section of E then*
   $$\nabla(fs) = f \cdot \nabla(s) + \lambda \cdot s \otimes df$$
   - *(b) For each $x \in D$ the homomorphism induced in the filtration over the fiber E_x satisfies*
   $$\nabla(E_{x,i}) \subseteq E_{x,i} \otimes K(D)|_x$$
   - *(c) For every $x \in D$ and every $i = 1, \dots, r$ the action of $\mathrm{Res}(\nabla, x)$ on $E_{x,i}/E_{x,i-1}$ is the multiplication by $\lambda\alpha_i(x)$. Since $\mathrm{Res}(\nabla, x)$ preserves the filtration, it acts on each quotient.*
   - *(d) The operator $\bigwedge^r E \longrightarrow (\bigwedge^r E) \otimes K(D)$ induced by ∇ coincides with $\lambda \cdot \nabla_\xi$.*

We denote by $\mathcal{M}_{Hod}(r,\alpha,\xi)$ the moduli space of triples of the form (λ, E, ∇), where λ is a complex number and (E,∇) is a parabolic λ-connection. The existence of this moduli was proved in [IIS06, §5]. The moduli space $\mathcal{M}_{Hod}(r,\alpha,\xi)$ is a complex algebraic variety of dimension $1 + 2(g-1)(r^2-1) + n(r^2 - r)$. It is equipped with a surjective algebraic morphism

$$\mathrm{pr}_\lambda : \mathcal{M}_{Hod}(r,\alpha,\xi) \longrightarrow \mathbb{C} \tag{9.3.1}$$

defined by $\mathrm{pr}_\lambda(\lambda, E, \nabla) = \lambda$.

Given a parabolic vector bundle $E \in \mathcal{M}(r,\alpha,\xi)$, taking $\lambda = 0$, a parabolic 0-connection over E is a homomorphism $\nabla : E \to E \otimes K(D)$ that preserves the filtration and such that for every $x \in D$, $\mathrm{Res}(\nabla, x)$ acts as the zero morphism on $E_{x,i}/E_{x,i+1}$. Then, for every $x \in D$, $\nabla(E_{x,i}) \subseteq E_{x,i+1} \otimes K(D)|_x$. Moreover, the induced morphism $\bigwedge^r E \to \bigwedge^r E \otimes K(D)$ is zero. As this morphism coincides locally with the trace of ∇, ∇ is a traceless morphism $\nabla : E \to E \otimes K(D)$.

Thus, a 0-connection is a traceless strictly parabolic Higgs bundle, so

$$\mathcal{M}_{Higgs}(r,\alpha,\xi) = pr_\lambda^{-1}(0) \subset \mathcal{M}_{Hod}(r,\alpha,\xi)$$

In particular, the embedding (9.2.1) of $\mathcal{M}(r,\alpha,\xi)$ into $\mathcal{M}_{Higgs}(r,\alpha,\xi)$ also gives an embedding of $\mathcal{M}(r,\alpha,\xi)$ into $\mathcal{M}_{Hod}(r,\alpha,\xi)$

$$i : \mathcal{M}(r,\alpha,\xi) \hookrightarrow \mathcal{M}_{Hod}(r,\alpha,\xi) \tag{9.3.2}$$

$\mathbb{C}^*$ acts on $\mathcal{M}_{Hod}(r,\alpha,\xi)$ extending the $\mathbb{C}^*$ action on $\mathcal{M}_{Higgs}(r,\alpha,\xi)$ introduced in formula (9.2.9) by

$$t \cdot (\lambda, E, \nabla) = (t \cdot \lambda, E, t \cdot \nabla) \tag{9.3.3}$$

Proposition 9.3.2. *Let Z be an irreducible component of the fixed point locus of the action 9.3.3 in $\mathcal{M}_{Hod}(r,\alpha,\xi)$. Then $\dim(Z) \le (r^2-1)(g-1) + \frac{n(r^2-r)}{2}$, with equality only for $Z = i(\mathcal{M}(r,\alpha,\xi))$*

Proof. A point $(\lambda, E, \nabla) \in \mathcal{M}_{Hod}(r,\alpha,\xi)$ can only be fixed if $\lambda = 0$. Then $Z \subseteq \mathcal{M}_{Higgs}(r,\alpha,\xi)$. The result follows from proposition 9.2.6. □

A 1-connection in $\mathcal{M}_{Hod}(r,\alpha,\xi)$ is a holomorphic connection on a parabolic vector bundle in the usual way [Bis02, §3], that is, a logarithmic connection singular over D such that the residue at every $x \in D$ restricted to $E_{x,i}/E_{x,i-1}$ is just the multiplication by $\alpha_i(x)$, so

$$\mathcal{M}_{conn}(r,\alpha,\xi) := \mathrm{pr}_\lambda^{-1}(1) \subset \mathcal{M}_{Hod}(r,\alpha,\xi)$$

is the moduli space of parabolic $SL(r,\mathbb{C})$-connections (E,∇) with weight system α and an isomorphism $\det(E) \cong \xi$. We denote by

$$\mathcal{M}^{st}_{conn}(r,\alpha,\xi) \subset \mathcal{M}_{conn}(r,\alpha,\xi) \quad \text{and} \quad \mathcal{M}^{st}_{Hod}(r,\alpha,\xi) \subset \mathcal{M}_{Hod}(r,\alpha,\xi)$$

the Zariski open subvarieties where the underlying parabolic vector bundle is stable.

Proposition 9.3.3. *The forgetful map*

$$\mathrm{pr}_E : \mathcal{M}^{st}_{conn}(r,\alpha,\xi) \longrightarrow \mathcal{M}^{s}(r,\alpha,\xi) \tag{9.3.4}$$

defined by $\mathrm{pr}_E(E,\nabla) = E$ admits no holomorphic section.

Proof. Any two parabolic holomorphic $SL(r,\mathbb{C})$-connections on E differ by a parabolic Higgs field $\Phi : E \to E \otimes K(D)$ with trace $\mathrm{tr}(\Phi) = 0$ and the same weight system. Taking into account the isomorphism (9.2.3), the map pr_E in (9.3.4) is a holomorphic torsor under the holomorphic cotangent bundle $T^*\mathcal{M}^s(r,\alpha,\xi) \to \mathcal{M}^s(r,\alpha,\xi)$.

Let $\rho : Y \to X$ be a cyclic Galois-covering totally ramified over D with big enough degree (see [Fal93, §V] for the details). Let $\mathcal{M}^s_\rho(r,\xi) \subseteq \mathcal{M}^s(r,\xi)$ be the moduli space of Galois-equivariant stable holomorphic vector bundles of rank r with an isomorphism $\det(E) \cong \xi$. There exists a isomorphism between $\mathcal{M}^s_\rho(r,\xi)$ and the moduli space of parabolic vector bundles over X singular over D of rank r with an isomorphism $\det(E) \cong \xi$.

Let $\mathcal{M}^s_{par}(r,\xi,D)$ be the moduli of parabolic vector bundles on X singular over D of rank r and $\det(E) \cong \xi$. By [Fal93, V.(vi)] there exists a line bundle $\mathcal{L}$ over $\mathcal{M}^s_{par}(r,\xi,D)$ with fibers $\det(H^1(Y,\mathrm{ad}(E)))$ such that the moduli of parabolic holomorphic connections over stable parabolic vector bundles is isomorphic to the torsor of holomorphic connections on $\mathcal{L}$.

Let $\mathcal{F}$ be the restriction of $\mathcal{L}$ to the subvariety $\mathcal{M}^s(r,\alpha,\xi) \subseteq \mathcal{M}^s_{par}(r,\xi,D)$. We obtain that the torsor of holomorphic connections over $\mathcal{F}$ is isomorphic to $\mathcal{M}^{st}_{conn}(r,\alpha,\xi)$. Following the ideas in [BGH13, Proposition 4.4], we will prove that $\mathcal{F}$ is ample, so its first chern class is nonzero and so $\mathcal{F}$ admits no holomorphic connections.

Using the analogous correspondence for the non-parabolic situation [Fal93, Lemma IV.4], there exists a line bundle $\mathcal{G}$ over $\mathcal{M}^s(r,\xi)$ with fibers $\det(H^1(Y,adE))$, such that $\mathcal{M}_{conn}(r,\xi)$ is isomorphic to the torsor of holomorphic connections on $\mathcal{L}$. By construction of the Galois-covering, we get the chain of inclusions

$$\begin{tikzcd} \mathcal{F} \arrow[d] & \mathcal{L} \arrow[d] & & \mathcal{G} \arrow[d] \\ \mathcal{M}^s(r,\alpha,D) \arrow[r, hook] & \mathcal{M}^s_{par}(r,\xi,D) \arrow[r, phantom, "\cong"] & \mathcal{M}^s_\rho(r,\xi) \arrow[r, hook] & \mathcal{M}^s(r,\xi) \end{tikzcd} \tag{9.3.5}$$

By construction of $\mathcal{G}$ and $\mathcal{L}$, $\mathcal{L}$ is the pullback of $\mathcal{G}$ by chain of morphisms in (9.3.5), and as $\mathcal{F}$ is its restriction to $\mathcal{M}^s(r,\alpha,D)$, $\mathcal{F}$ is the pullback of $\mathcal{G}$ by the given chain of morphisms. $\mathcal{G}$ is ample [BGH13, Proposition 4.4] and the restriction of an ample line bundle is ample, so $\mathcal{F}$ must be ample. □

The forgetful maps (9.3.4) and (9.2.2) can be both seen as restrictions to $\mathrm{pr}_\lambda^{-1}(0)$ and $\mathrm{pr}_\lambda^{-1}(1)$ respectively of a map

$$\mathrm{pr}_E : \mathcal{M}^{st}_{Hod}(r,\alpha,\xi) \longrightarrow \mathcal{M}^s(r,\alpha,\xi) \tag{9.3.6}$$

defined by $\mathrm{pr}_E(\lambda,E,\nabla) = E$.

Corollary 9.3.4. *The only holomorphic map*

$$s : \mathcal{M}^s(r,\alpha,\xi) \longrightarrow \mathcal{M}^{st}_{Hod}(r,\alpha,\xi)$$

with $\mathrm{pr}_E \circ s = id$ *is the restriction of the embedding* i *defined in (9.3.2)*

$$i : \mathcal{M}^s(r,\alpha,\xi) \hookrightarrow \mathcal{M}^{st}_{Hod}(r,\alpha,\xi)$$

Proof. The composition

$$\mathcal{M}^s(r,\alpha,\xi) \xrightarrow{s} \mathcal{M}^{st}_{Hod}(r,\alpha,\xi) \xrightarrow{\mathrm{pr}_\lambda} \mathbb{C}$$

is a holomorphic function on $\mathcal{M}^s(r,\alpha,\xi)$. Since the later is compact, it is a constant function. Up to the $\mathbb{C}^*$ action in (9.3.3), we may assume that this constant is either 0 or 1.

If this constant were 1, then s would factor through $\mathrm{pr}_\lambda^{-1}(1) = \mathcal{M}^{st}_{conn}(r,\alpha,\xi)$, which would contradict Proposition 9.3.3. Hence this constant is 0, and s factors through $\mathrm{pr}_\lambda^{-1}(0) = \mathcal{M}^{st}_{Higgs}(r,\alpha,\xi)$. Thus, under isomorphism (9.2.4) s corresponds to a holomorphic global section of $T^*\mathcal{M}^s(r,\alpha,\xi)$. But due to Lemma 9.2.3, s vanishes, so it must be the restriction of the canonical embedding i in (9.3.2). □

Corollary 9.3.5. *Let $\mathcal{M}^{sm}_{Hod}(r,\alpha,\xi)$ be the smooth locus of $\mathcal{M}_{Hod}(r,\alpha,\xi)$. The restriction of the holomorphic tangent bundle*

$$T\mathcal{M}^{sm}_{Hod}(r,\alpha,\xi) \longrightarrow \mathcal{M}^{sm}_{Hod}(r,\alpha,\xi)$$

to $i(\mathcal{M}^s(r,\alpha,\xi)) \subset \mathcal{M}^{sm}_{Hod}(r,\alpha,\xi)$ does not admit any nonzero holomorphic section.

Proof. Let $\mathcal{N}$ be the holomorhpic normal bundle of the restricted embedding

$$i : \mathcal{M}^s(r,\alpha,\xi) \hookrightarrow \mathcal{M}^{sm}_{Hod}(r,\alpha,\xi)$$

Due to Lemma 9.2.3, it suffices to show that this vector bundle $\mathcal{N}$ over $\mathcal{M}^s(r,\alpha,\xi)$ has no nonzero holomorphic sections. One has a canonical isomorphism

$$\mathcal{M}^{sm}_{Hod}(r,\alpha,\xi) \xrightarrow{\sim} \mathcal{N} \tag{9.3.7}$$

of varieties over $\mathcal{M}^s(r,\alpha,\xi)$, defined by sending any (λ, E, ∇) to the derivative at $t = 0$ of the map $\mathbb{C} \longrightarrow \mathcal{M}_{Hod}(r,\alpha,\xi)$ given by

$$t \longmapsto (t\cdot\lambda, E, t\cdot\nabla)$$

Using this morphism, from Corollary 9.3.4 we conclude that $\mathcal{N}$ does not have any nonzero holomorphic sections. □

Corollary 9.3.6. *Let $r = 2$. The isomorphism class of the complex analytic space $\mathcal{M}_{Hod}(r,\alpha,\xi)$ determines uniquely the isomorphism class of the punctured Riemann surface (X, D).*

Proof. We will proceed similarly to the proof of Corollary 9.2.7. Let $Z \subset \mathcal{M}_{Hod}(r,\alpha,\xi)$ be a closed analytic subset with the following three properties:

1. Z is irreducible and has complex dimension $(r^2-1)(g-1) + \frac{n(r^2-r)}{2}$.
2. The smooth locus $Z^{sm} \subseteq Z$ lies in the smooth locus $\mathcal{M}^{sm}_{Hod}(r,\alpha,\xi) \subset \mathcal{M}_{Hod}(r,\alpha,\xi)$
3. The restriction of the holomorphic tangent bundle $T\mathcal{M}^{sm}_{Hod}(r,\alpha,\xi)$ to the subspace $Z^{sm} \subset \mathcal{M}^{sm}_{Hod}(r,\alpha,\xi)$ has no nonzero holomorphic sections.

By Corollary 9.3.5, $i(\mathcal{M}(r,\alpha,\xi)) \subset \mathcal{M}_{Hod}(r,\alpha,\xi)$ of the embedding (9.3.2) has these properties. We will prove that this is the only possible choice for Z.

Every $\mathbb{C}^*$ action on $\mathcal{M}_{Hod}(r,\alpha,\xi)$ defines a holomorphic vector field on its smooth locus. The third assumption on Z says that any such holomorphic vector field vanishes on Z^{sm}. Therefore, the stabilizer of each point $Z^{sm} \subset \mathcal{M}_{Hod}^{sm}(r,\alpha,\xi)$ has nontrivial tangent space at $1 \in \mathbb{C}^*$, and hence the stabilizer must be the full group $\mathbb{C}^*$.

Then Z^{sm} belongs to the fixed point locus of the action (9.3.3) in $\mathcal{M}_{Hod}(r,\alpha,\xi)$, and thus, so does its closure in $\mathcal{M}_{Hod}(r,\alpha,\xi)$, Z. Due to Proposition 9.3.2, and property (1), $Z = i(\mathcal{M}(r,\alpha,\xi))$. In particular, we have $Z \cong \mathcal{M}(r,\alpha,\xi)$.

Then, as in the proof of corollary 9.2.7, the isomorphism class of $\mathcal{M}_{Higgs}(r,\alpha,\xi)$ determines the isomorphism class of $\mathcal{M}(r,\alpha,\xi)$. Due to [BdBnB01, Theorem 3.2], the latter determines the isomorphism class of the punctured Riemann surface (X, D). □

The rank two condition of the previous corollary is only necessary in order to apply the Torelli theorem in [BdBnB01]. If the theorem of [BdBnB01] were extended to Higher rank, then corollary 9.3.6 would also hold for higher rank with the same proof given above.

9.4 The parabolic Deligne-Hitchin moduli space

We can extend Deligne's construction [Del89] of the Deligne-Hitchin moduli space for the group $\mathrm{SL}(r,\mathbb{C})$ as described in [BGHL09, 4] to the parabolic scenario. Let α be a system of weights over D such that for every $x \in D$,

$$\sum_{i=1}^{r} \alpha_i(x) \in \mathbb{Z} \tag{9.4.1}$$

Let ξ be the line bundle over X given by $\xi = \mathcal{O}_X\left(\sum_{x\in D}\left(\sum_{i=1}^r \alpha_i(x)\right)x\right)$. Let $X_{\mathbb{R}}$ be the C^∞ real manifold of dimension two underlying X. Fix a point $x_0 \in X_{\mathbb{R}}\backslash D$. For every $x \in D$, let $\gamma_x \in \pi_1(X_{\mathbb{R}}\backslash D, x_0)$ be the class of a loop around x. Let $\mathcal{M}_{rep}(X_{\mathbb{R}}, r, \alpha)$ be the subvariety of $\mathrm{Hom}(\pi_1(X_{\mathbb{R}}\backslash D, x_0), \mathrm{SL}(r,\mathbb{C}))/\!\!/\,\mathrm{SL}(r,\mathbb{C})$ corresponding to classes of representations $\rho : \pi_1(X_{\mathbb{R}}\backslash D, x_0) \longrightarrow \mathrm{SL}(r,\mathbb{C})$ such that for each $x \in D$, $\rho(\gamma_x)$ has eigenvalues $\{e^{-2\pi i\alpha_i(x)}\}$. The group $\mathrm{SL}(r,\mathbb{C})$ acts on $\mathrm{Hom}(\pi_1(X_{\mathbb{R}}\backslash D, x_0), \mathrm{SL}(r,\mathbb{C}))$ through the adjoint action of $\mathrm{SL}(r,\mathbb{C})$ on itself. Since the eigenvalues of $\rho(\gamma_x)$ are preserved by conjugation, the quotient is well defined. On the other hand, the determinant of $\rho(\gamma_x)$ is the product of its eigenvalues, so

$$\det(\rho(\gamma_x)) = \prod_i e^{-2\pi i\alpha_i(x)} = e^{-2\pi i\sum_i \alpha_i(x)} = 1$$

The fundamental groups for different base points are identified up to an inner automorphism and the different choices of the loops γ_x are identified through an outer isomorphism. Thus, the isomorphism class of the space $\mathcal{M}_{rep}(X_{\mathbb{R}}, r, \alpha)$ is independent of the choice of x_0 and the loops γ_x, so we can omit any reference to both of them.

A Riemann-Hilbert-like correspondence can be defined for the moduli spaces of parabolic connections and the moduli space of filtered local systems [Sim90]. [IIS06] and [Sim90] imply that there exist a biholomorphic isomorphism

$$\mathcal{M}_{rep}(X_{\mathbb{R}}, r, \alpha) \xrightarrow{\sim} \mathcal{M}_{con}(r, \alpha, \xi) \tag{9.4.2}$$

The isomorphism sends each representation $\rho : \pi_1(X_{\mathbb{R}} \backslash D, x_0) \longrightarrow SL(r, \mathbb{C})$ to an associated parabolic $SL(r, \mathbb{C})$-bundle $E_X(\rho)$ over X with weight system α, endowed with a parabolic connection $\nabla_X(\rho)$.

Composing the isomorphism (9.4.2) with the action of $\mathbb{C}^*$ in the moduli space of parabolic λ-connections given by (9.3.3) gives us an embedding of $\mathcal{M}_{rep}(X_{\mathbb{R}}, \alpha) \hookrightarrow \mathrm{pr}_\lambda^{-1}(\lambda)$ for every $\lambda \in \mathbb{C}^*$. This defines a holomorphic open embedding

$$\mathbb{C}^* \times \mathcal{M}_{rep}(X_{\mathbb{R}}, r, \alpha) \hookrightarrow \mathcal{M}_{Hod}(r, \alpha, \xi, X) \tag{9.4.3}$$

onto the open locus $\mathrm{pr}_\lambda^{-1}(\mathbb{C}^*) \subset \mathcal{M}_{Hod}(r, \alpha, \xi, X)$.

Let J_X denote the almost complex structure of the Riemann surface X. Then $-J_X$ is also an almost complex structure on $X_{\mathbb{R}}$. The Riemann surface defined by $-J_X$ wil be denoted by $\overline{X}$. Similarly, let $\overline{\xi}$ be the vector bundle obtained with the conjugate almost complex structure of ξ.

We can also consider the moduli space $\mathcal{M}_{Hod}(r, \alpha, \overline{\xi}, \overline{X})$ of parabolic λ-connections on $\overline{X}$, etcetera. Now, we define the parabolic Deligne-Hitchin moduli space

$$\mathcal{M}_{DH}(r, \alpha, \xi, X) := \mathcal{M}_{Hod}(r, \alpha, \xi, X) \cup \mathcal{M}_{Hod}(r, \alpha, \overline{\xi}, \overline{X})$$

by gluing $\mathcal{M}_{Hod}(r, \alpha, \xi, X)$ to $\mathcal{M}_{Hod}(r, \alpha, \overline{\xi}, \overline{X})$ along the image of $\mathbb{C}^* \times \mathcal{M}_{rep}(X_{\mathbb{R}}, r, \alpha)$ for the map in (9.4.3). More precisely, we identify, for each $\lambda \in \mathbb{C}^*$ and each representation $\rho \in \mathcal{M}_{rep}(X_{\mathbb{R}}, r, \alpha)$, the two points

$$(\lambda, E_X(\rho), \lambda \cdot \nabla_X(\rho)) \in \mathcal{M}_{Hod}(r, \alpha, \xi, X) \text{ and } (\lambda^{-1}, E_{\overline{X}}(\rho), \lambda^{-1} \cdot \nabla_{\overline{X}}(\rho)) \in \mathcal{M}_{Hod}(r, \alpha, \overline{\xi}, \overline{X})$$

The forgetful map pr_λ in (9.3.6) extends to a natural holomorphic morphism

$$\mathrm{pr} : \mathcal{M}_{DH}(r, \alpha, \xi, X) \longrightarrow \mathbb{CP}^1 \tag{9.4.4}$$

whose fiber over $\lambda \in \mathbb{CP}^1$ is canonically biholomorphic to

- the moduli space $\mathcal{M}_{Higgs}(r, \alpha, \xi, X)$ of parabolic $SL(r, \mathbb{C})$ Higgs bundles on X of weight system α and $\det(E) \cong \xi$ if $\lambda = 0$
- the moduli space $\mathcal{M}_{Higgs}(r, \alpha, \overline{\xi}, \overline{X})$ of parabolic $SL(r, \mathbb{C})$ Higgs bundles on $\overline{X}$ of weight system α and $\det(E) \cong \overline{\xi}$ if $\lambda = \infty$
- the moduli space of parabolic λ-connections on X of weight system α and $\det(E) \cong \xi$ for every fixed $\lambda \neq 0$ and $\lambda \neq \infty$. This fibres are also biholomorphic to the moduli space $\mathcal{M}_{rep}(X_{\mathbb{R}}, r, \alpha)$ of equivalence classes of representations $[\rho] \in \mathrm{Hom}(\pi_1(X_{\mathbb{R}} \backslash D, x_0), SL(r, \mathbb{C})) /\!\!/ SL(r, \mathbb{C})$ such that for some fixed loops $\gamma_x \in \pi_1(X_{\mathbb{R}} \backslash D, x_0)$ around the points $x \in D$, $\rho(\gamma_x)$ has eigenvalues $\{e^{-2\pi i \alpha_i(x)}\}$.

Now we can prove the main result.

Theorem 9.4.1. *Let $r = 2$. The isomorphism class of the complex analytic space $\mathcal{M}_{DH}(r, \alpha, \xi, X)$ determines uniquely the isomorphism class of the unordered pair of punctured Riemann surfaces $\{(X, D), (\overline{X}, D)\}$.*

Proof. We denote by $\mathcal{M}^{sm}_{DH}(r, \alpha, \xi, X)$ the smooth locus of $\mathcal{M}_{DH}(r, \alpha, \xi, X)$, and by

$$T\mathcal{M}^{sm}_{DH}(r, \alpha, \xi, X) \longrightarrow \mathcal{M}^{sm}_{DH}(r, \alpha, \xi, X)$$

its holomorphic tangent bundle. Since $\mathcal{M}_{Hod}(r, \alpha, \xi, X)$ is open in $\mathcal{M}_D H(r, \alpha, \xi, X)$, Corollary 9.3.5 implies that the restriction of $T\mathcal{M}^{sm}_{DH}(r, \alpha, \xi, X)$ to

$$i(\mathcal{M}^s(r, \alpha, \xi, X)) \subset \mathcal{M}^{sm}_{Hod}(r, \alpha, \xi, X) \subset \mathcal{M}^{sm}_{DH}(r, \alpha, \xi, X)$$

does not admit any nonzero holomorphic section. The same argument applies if we replace X by $\overline{X}$. Since $\mathcal{M}_{Hod}(r, \alpha, \overline{\xi}, \overline{X})$ is also open in $\mathcal{M}_{DH}(r, \alpha, \xi, X)$, the restriction of $T\mathcal{M}^{sm}_{DH}(r, \alpha, \xi, X)$ to

$$i(\mathcal{M}^s(r, \alpha, \overline{\xi}, \overline{X})) \subset \mathcal{M}^{sm}_{Hod}(r, \alpha, \overline{\xi}, \overline{X}) \subset \mathcal{M}^{sm}_{DH}(r, \alpha, \xi, X)$$

does not admit any nonzero holomorphic section either. We will extend the $\mathbb{C}^*$ action on $\mathcal{M}_{Hod}(r, \alpha, \xi, X)$ in (9.3.3) to $\mathcal{M}_{DH}(r, \alpha, \xi, X)$. We consider the corresponding $\mathbb{C}^*$ action on $\mathcal{M}_{Hod}(r, \alpha, \xi, \overline{X})$. The action of any $t \in \mathbb{C}^*$ on the open subset $\mathbb{C}^* \times \mathcal{M}_{rep}(X_{\mathbb{R}}, r, \alpha) \longrightarrow \mathcal{M}_{Hod}(r, \alpha, \xi, X)$ in (9.4.3) coincides with the action of $1/t$ on $\mathbb{C}^* \times \mathcal{M}_{rep}(X_{\mathbb{R}}, r, \alpha) \longrightarrow \mathcal{M}_{Hod}(r, \alpha, \overline{\xi}, \overline{X})$. Therefore, we get an action of $\mathbb{C}^*$ on $\mathcal{M}_{DH}(r, \alpha, \xi, X)$.

Due to Proposition 9.3.2, each irreducible component of the fixed point locus of this $\mathbb{C}^*$ action on $\mathcal{M}_{DH}(r, \alpha, \xi, X)$ has dimension less or equal to $(r^2 - 1)(g - 1) + \frac{n(r^2-r)}{2}$, with equality only for $i(\mathcal{M}(r, \alpha, \xi, X))$ and for $i(\mathcal{M}(r, \alpha, \overline{\xi}, \overline{X}))$.

In a similar way of the proof of Corollary 9.3.6, these observations imply that $\mathcal{M}_{DH}(r, \alpha, \xi, X)$ determines the isomorphism class of the unordered pair of moduli spaces $\{\mathcal{M}(r, \alpha, \xi, X), \mathcal{M}(r, \alpha, \overline{\xi}, \overline{X})\}$. Therefore, using [BdBnB01, Theorem 3.2] the statement of the theorem follows. $\square$

The rank two condition of the previous theorem is only necessary in order to apply the Torelli theorem in [BdBnB01]. If the theorem of [BdBnB01] were extended to Higher rank, then theorem 9.4.1 would also hold for higher rank with the same proof given above.

Glossary of Notations

$\boldsymbol{L(D)}$	Line bundle associated to the Cartier divisor D, definition 1.3.4, p. 16
$\mathbf{Pic}(\boldsymbol{X})$	Piccard variety of X, section 2.3.1, p.34
$\mathbf{Pic}(\boldsymbol{X})^{\boldsymbol{d}}$	Piccard variety of X of degree d, section 2.3.1, p. 34
$\mathbf{Jac}(\boldsymbol{X})$	Jacobian variety of X, section 2.3.1, p. 35
$\boldsymbol{\mu(E)}$	Slope of the vector bundle E, definition 2.4.1, p. 36
$\boldsymbol{\mathcal{M}(r,d)}$	Same as $\mathcal{M}(r,d,X)$. Moduli space of semistable vector bundles of rank r and degree d over X, section 2.4.2, p. 41
$\boldsymbol{\mathcal{M}^s(r,d)}$	Same as $\mathcal{M}^s(r,d,X)$. Moduli space of stable vector bundles of rank r and degree d over X, section 2.4.2, p. 42
$\boldsymbol{\mathcal{M}(r,\xi)}$	Same as $\mathcal{M}(r,\xi,X)$. Moduli space of semistable vector bundles of rank r and determinant ξ over X, section 2.4.2, p. 42
$\boldsymbol{\mathcal{M}^s(r,\xi)}$	Same as $\mathcal{M}^s(r,\xi,X)$. Moduli space of stable vector bundles of rank r and determinant ξ over X, section 2.4.2, p. 42
$\boldsymbol{\mathcal{M}_{Higgs}(r,d)}$	Same as $\mathcal{M}_{Higgs}(r,d,X)$ Moduli space of semistable Higgs bundles of rank r and degree d over X, section 3.2, p. 47
$\boldsymbol{\mathcal{M}_{Higgs}(r,\xi)}$	Same as $\mathcal{M}_{Higgs}(r,\xi,X)$. Moduli space of semistable traceless Higgs bundles of rank r and determinant ξ over X, section 3.2, p. 47
$\boldsymbol{\mathcal{M}^s_{Higgs}(r,d)}$	Same as $\mathcal{M}^s_{Higgs}(r,d,X)$ Moduli space of stable Higgs bundles of rank r and degree d over X, section 3.2, p. 47
$\boldsymbol{\mathcal{M}^s_{Higgs}(r,\xi)}$	Same as $\mathcal{M}^s_{Higgs}(r,\xi,X)$. Moduli space of stable traceless Higgs bundles of rank r and determinant ξ over X, section 3.2, p. 47
$\boldsymbol{\mathcal{M}^{st}_{Higgs}(r,\xi)}$	Same as $\mathcal{M}^{st}_{Higgs}(r,\xi,X)$. Locus of Higgs bundles in $\mathcal{M}_{Higgs}(r,\xi,X)$ whose underying vector bundle is stable, section 3.2, p. 48
$\mathbf{pardeg}(\boldsymbol{E})$	Parabolic degree of E, definition 4.2.1, p. 56

$\mathbf{par}\,\boldsymbol{\mu(E)}$	Parabolic slope of E, definition 4.2.1, p.57
$\boldsymbol{\mathcal{M}(r,d,\alpha)}$	Same as $\mathcal{M}(r,d,\alpha,X)$. Moduli space of semistable parabolic vector bundles of rank r, degree d and weight system α over X, section 4.3, p. 60
$\boldsymbol{\mathcal{M}^s(r,d,\alpha)}$	Same as $\mathcal{M}^s(r,d,\alpha,X)$. Moduli space of stable parabolic vector bundles of rank r, degree d and weight system α over X, section 4.3, p. 60
$\boldsymbol{\mathcal{M}(r,\alpha,\xi)}$	Same as $\mathcal{M}(r,\alpha,\xi,X)$. Moduli space of semistable parabolic vector bundles of rank r, determinant ξ and weight system α over X, section 4.3, p. 60
$\boldsymbol{\mathcal{M}^s(r,\alpha,\xi)}$	Same as $\mathcal{M}^s(r,\alpha,\xi,X)$. Moduli space of stable parabolic vector bundles of rank r, determinant ξ and weight system α over X, section 4.3, p. 60
$\mathbf{SParEnd}\boldsymbol{(E)}$	Sheaf of strongly parabolic endomorphisms of E, section 5.1, p. 62
$\mathbf{ParEnd}\boldsymbol{(E)}$	Sheaf of non-strongly parabolic endomorphisms of E, section 5.1, p. 62
$\boldsymbol{\mathcal{M}_{Higgs}(r,\alpha,\xi)}$	Same as $\mathcal{M}_{Higgs}(r,\alpha,\xi,X)$. Moduli space of semistable traceless strongly parabolic Higgs bundles of rank r, determinant ξ and weight system α over X, section 5.3, p. 64
$\boldsymbol{\mathcal{M}^{sm}_{Higgs}(r,\alpha,\xi)}$	Smooth locus of $\mathcal{M}_{Higgs}(r,\alpha,\xi)$, section 5.3, p. 64
$\boldsymbol{\mathcal{M}^{ns}_{Higgs}(r,\alpha)}$	Moduli space of non-strictly parabolic semistable Higgs bundles of rank r and weight system α over X, section 5.3, p. 64
$\boldsymbol{\mathcal{M}^{st}_{Higgs}(r,\alpha,\xi)}$	Locus of parabolic Higgs bundles in $\mathcal{M}_{Higgs}(r,\alpha,\xi)$ whose underying parabolic vector bundle is stable, section 5.3, p. 65
$\mathbf{Res}\boldsymbol{(\nabla,x)}$	Residue of the parabolic connection ∇ at the point x, section 6.1.1, p. 68
$\boldsymbol{\mathcal{M}_{rep}(X_{\mathbb{R}},\mathbf{GL}(r,\mathbb{C}))}$	Moduli space of representations of the fundamental group of X in $\mathrm{GL}(r,\mathbb{C})$, section 6.1.2, p. 70
$\boldsymbol{\mathcal{M}_{conn}(r,\alpha,\xi)}$	Same as $\mathcal{M}_{conn}(r,\alpha,\xi,X)$. Moduli of parabolic connections with determinant ξ and system of weights α on X for the group $\mathrm{SL}(r,\mathbb{C})$, section 6.3, p. 74
$\boldsymbol{\mathcal{M}_{Hod}(r,X)}$	Same as $\mathcal{M}_{Hod}(X,\mathcal{L}(r,\mathbb{C}))$. Hodge moduli space of X of rank r, definition 8.1.1, p. 82
$\boldsymbol{\mathcal{M}_{DH}(r,X)}$	Deligne-Hitchin moduli space of X of rank r, definition definition 8.2.1, p. 84

$\boldsymbol{\mathcal{M}_{Hod}(r,\alpha,\xi)}$	Same as $\mathcal{M}_{Hod}(r,\alpha,\xi,X)$. Parabolic hodge moduli space of X of rank r, determinant ξ and system of weights α, definition 9.3.1, p. 98
$\boldsymbol{\mathcal{M}^{st}_{conn}(r,\alpha,\xi)}$	Locuss of parabolic connections in $\mathcal{M}_{conn}(r,\alpha,\xi)$ such that its underying parabolic vector bundle is stable, section 9.3, p. 99
$\boldsymbol{\mathcal{M}^{st}_{Hod}(r,\alpha,\xi)}$	Locuss of parabolic λ-connections in $\mathcal{M}_{Hod}(r,\alpha,\xi)$ such that its underying parabolic vector bundle is stable, section 9.3, p. 99
$\boldsymbol{\mathcal{M}^{sm}_{Hod}(r,\alpha,\xi)}$	Smooth locus of $\mathcal{M}_{Hod}(r,\alpha,\xi)$, corollary 9.3.5, p. 101
$\boldsymbol{\mathcal{M}_{rep}(X_{\mathbb{R}},r,\alpha)}$	Moduli space of $\mathrm{SL}(r,\mathbb{C})$ representations for the weight system α, section 9.4, p.103
$\boldsymbol{\mathcal{M}_{DH}(r,\alpha,\xi,X)}$	Parabolic Deligne-Hitchin moduli space of X with rank r, determinant ξ and weight system α, section 9.4, p. 103
$\boldsymbol{\mathcal{M}^{sm}_{DH}(r,\alpha,\xi,X)}$	Smooth locus of $\mathcal{M}_{DH}(r,\alpha,\xi,X)$, theorem 9.4.1, p. 104

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